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THE SCIENCE OF ARITHMETIC,

FOR

HIGH SCHOOLS, NORMAL SCHOOLS, PREPARATORY DEPARTMENTS TO COLLEGES, AND ACADEMIES.

By EDWARD OLNEY,

PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF MICHIGAN, AND AUTHOR OF A SERIES OF MATHEMATICAL TEXT-BOOKS.

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PREFACE.

THIS is not an arithmetic for beginners; the author's ELEMENTS OF ARITHMETIC, or some similar work, is presumed to precede it in the course. This is designed for pupils who can push their studies beyond the mere rudiments of a common English education, as, for example, through a High School, a Normal, or an Academic course, or who propose a College course.

Some of the purposes which have guided the author in the preparation of the work are,

1. To have every section, even the very first, fresh and new, both in matter and method, to a student who has a good knowledge of the elements of arithmetic, while at the same time all needed review of Elementary Arithmetic is secured.

2. To give such a comprehensive and philosophical view of arithmetical principles and processes, as the broader scholarship of this grade of students demands, and as will enable the student to retain the subject as a whole.

3. To prepare the way for future mathematical studies, by familiarizing the pupil with the elements of the literal notation and the use of formulæ—the very alphabet of all mathematical study above the merest rudiments of arithmetic.

4. To give a large amount of practical information, especially on business arithmetic, which it is desirable for a liberally educated person, or a teacher, to possess.

5. To train the student to habits of clear and accurate thought, and to the expression of such thought in perspicuous and elegant language.

In accordance with these purposes, the reader will observe that the chapter on Notation contains much matter that will be new and interesting to the student of Elementary Arithmetic, and will call for vigorous thought. While the subjects of Elementary Arithmetic are involved, so that the student must recall them, and

refresh his mind if need be, they do not form the staple of any part of the chapters covering the ground of the Fundamental Rules, Fractions and Reduction, as ordinarily treated. Following the chapter on Notation, we have one on Reduction, in which it is shown how all forms of arithmetical reduction, as the various reductions of fractions, common and decimal, reductions of compound numbers, etc., are included under one simple principle, in the application and illustration of which all the pupil's knowledge of these subjects as presented in elementary arithmetic is required, and virtually reviewed, while at the same time he is learning something new, and getting more broad and liberal views of the subject. The very place given to Reduction—immediately after Notation and before Addition, Subtraction, etc.—will arrest attention and lead to thought. Of course, such an arrangement would be impracticable in a first book in arithmetic; but that it is philosophical, there can be no question. Reduction is but a change of notation, and hence the consideration of it comes logically in immediate connection with notation.

The chapters on Combinations and Resolutions of Numbers, including Addition, Multiplication, Subtraction, and Division, are designed to exhibit the fact that in each of these processes the fundamental principles are the same, whether we combine simple or compound numbers, integral or fractional, or whether we use the Arabic or the literal notation. Moreover, new views of the relations of these operations are suggested by the order of arrangement, as well as by the detailed treatment. Here, again, while each page will be found to contain matter which is fresh and new, a thorough review of the principles and processes of the Fundamental Rules as applied to simple and compound numbers and fractions, will be secured.

Again, it will be observed that *scarcely any rules are given.* This is not that the author thinks them unnecessary—he considers it essential that a pupil should be able to give promptly, and elegantly, a rule for every important arithmetical process, and also a good logical demonstration of it; but, *at this stage of his advancement*, it is thought that, with much care on the part of the teacher, the pupil may be able to make his own rules, after having thoroughly mastered the underlying principles. Hence such directions are given instead of giving the rule. It is recommended

that each pupil have a blank book in which to record these rules after he has written them, and they have been fully criticised and amended as they may need. Unless great care is taken in this respect, the omission of rules will lead to slovenly habits of thought and expression; but with proper care, and the awakening of a spirit of *independent effort*, this exercise may be one of the most useful which the study will furnish.

A person opening this book may fall upon places that will make it look quite like Algebra; but only the mere elements of the literal notation are introduced, and the solution of the simple equation. A teacher who has never studied Algebra at all will have no difficulty in using this book, as all the principles required are fully developed, and in the most elementary manner. The design of this is to familiarize the student with the use of formulæ, without which nothing can be done in mathematics beyond the mere elements of arithmetic; or, at least, nothing can be done in an elegant and scientifically mathematical way. But, to the intelligent and enterprising teacher, the author feels that he needs make no apology for this feature of the work, and it is for the use of such teachers and schools that the book is written.

The broad and comprehensive treatment of subjects will be seen at a glance. In addition to what has already been said about Reduction, as in all its forms covered by a single principle, and of the Fundamental Rules, we may call attention to the treatment of Evolution, in which the pupil is taught to read the rule for the extraction of any root from the corresponding power of a binomial; to the problems in Percentage, in which a single principle is seen to cover all cases; to Equation of Payments and Averaging Accounts, where a single brief rule includes all cases; to Life Insurance,* where one line of thought gives the development of the subject in all its most important cases; and to other topics which will readily occur to the reader as illustrating the same method.

In treating Business Arithmetic the author has proceeded upon the presumption that the thing most needed by the pupil is a

* As to this subject, which so interests the great majority of our citizens, the author believes this to be the first attempt to present the elements of the subject in an arithmetic, in accordance with the principles actually used by actuaries. The presentation ordinarily made is so meager as to give no insight whatever into the underlying principles, or when attempts have been made tc present these principles, they have been misrepresented.

clear understanding of the nature of the transactions, the arithmetical processes being simple enough when the circumstances of the case are well understood. Hence more information concerning business matters is given than is customary, and great care has been taken to give the examples an air of actual life transactions. The demands of modern business have given rise to a large variety of problems in Discount, which have never been treated in our Arithmetics. In this respect it is thought this book will meet a felt want. Most of the great variety of cases herein treated are such as have come to the author for solution from business men, several of them having been received many times—they are not fictitious cases, as any practical business man will see. In fact, in all parts of the book, there has been a rigid exclusion of mere fictitious problems, and it is not impossible that some teachers observing the paucity of such examples as "If a wolf and a bear eat $\frac{3}{8}$ of a cow in $\frac{4}{5}$ of a day, etc.," may think that there has been a very great oversight. But the author would assure any such that he never knew a practical case of the kind to be presented for solution by a mathematician, and must think them trash.

In conclusion, it will be observed that the needs of a young person fitting himself for teaching have been kept constantly in view. The breadth of view given; the outlooks into new and interesting fields of thought; the great amount of practical information; the careful exhibition of the order of development of subjects; in Business Arithmetic, the double solutions of problems by the formulæ, and by the elementary analysis; and many other features will at once commend themselves as important features in a work designed for such students.

With grateful appreciation of the favor with which the other books of his mathematical series have been received by his fellow-teachers, and the public, the author submits this, the last written of the series, with the assurance that it will receive all the attention which its merits may demand.

EDWARD OLNEY.

ANN ARBOR, MICH., *April*, 1876.

This book may be had with or without the answers in the back part. A KEY is also published, which contains, in addition to the solutions, numerous practical suggestions to teachers, and all the rules.

CONTENTS.

HISTORICAL NOTE.—The present mathematical notation, except as regards the Arabic figures, is of quite recent European origin. The signs + and — were introduced in 1554 by Stifelius, a German, who also introduced the radical sign, $\sqrt{}$. The sign × was introduced by Oughtreed in 1661; ÷ by Braucker in 1668; and = in 1557 by Recorde;—all of England. The vinculum, ———, was introduced by Vieta, a Frenchman, about 1600, and in 1629 Gérard, a Flemish mathematician, suggested the parentheses, ( ). The literal notation is due to Vieta and Descartes, the use of coefficients to Scheubelius, a German (1550), and the present use of exponents had not been fully settled in Newton's time.

INTRODUCTION.

1. Pure Mathematics is a general term applied to several branches of science, which have for their object the investigation of the properties and relations of quantity—comprehending number, and magnitude as the result of extension—and of form.

2. The Several Branches of Pure Mathematics are Arithmetic, Algebra, Calculus, and Geometry.

3. Arithmetic, Algebra, and Calculus treat of number, and Geometry treats of magnitude as the result of extension.

4. Quantity is the amount or extent of that which may be measured; it comprehends number and magnitude.

The term quantity is also conventionally applied to symbols used to represent quantity. Thus 25, *m*, XI, etc., are called quantities, although, strictly speaking, they are only representatives of quantities.

It is not easy to give a philosophical account of the idea or ideas, represented by the word *Quantity* as used in Mathematics; and doubtless, different persons use the word in somewhat different senses. It is obviously incorrect to say that "Quantity is anything which can be measured." Thus, a load of wood, or a piece of ground, can be measured; but no one would think of the wood or piece of

ground as being the quantity. The *quantity* (of wood or ground) is rather the *amount* or *extent* of it.

The word is very convenient as a general term for mathematical concepts,* when we wish to speak of them without indicating whether it is number or magnitude that is meant. Thus we say, "*m* represents a certain quantity," and do not care to be more specific; *i. e.*, do not care to say whether it is a number, a line, a surface, or a solid, that is meant.

As applied to number, perhaps the term conveys the idea of the whole, rather than of that whole as made up of parts. It is, therefore, scarcely proper to speak of multiplying by a *quantity;* we should say, by a number.

The distinction between quantity and number is marked by the questions, "How much?" and "How many?"

5. *Number* is quantity conceived (thought of) as made up of parts, and answers to the question, "How many?"

Thus, a distance is a quantity; but, if we call that distance 5, we convert the notion into number, by indicating that the distance under consideration is made up of parts. Now, the distance may be just the same, whether we consider it as a whole, or think of it as 5; *i. e.*, as made up of 5 equal parts. Again, *m* may mean a value, as of a farm. We may or may not conceive it as a number (as of dollars). If we think of it simply in the aggregate, as the worth of a farm, *m* represents quantity; if we think of it as made up of parts (as of dollars), it is a number.

[What is said about Discontinuous and Continuous Number, including the second definition of Arithmetic, can be omitted, if the teacher thinks best, without breaking the continuity of the subject.]

6. Number is of two kinds, ***Discontinuous*** and ***Continuous.***

7. *Discontinuous Number* is number conceived as made up of finite parts; or it is number which passes

* A *concept* is an abstract object of thought—a thought-object. The mathematical concepts are *number* (both continuous and discontinuous), *magnitude* (including lines, surfaces, and solids), and *form*.

from one state of value to another by the successive additions or subtractions of finite units; *i. e.*, units of appreciable magnitude.

8. *Continuous Number* is number which is conceived as composed of infinitesimal parts; or it is number which passes from one state of value to another by passing through all intermediate values, or states.

Number, as the pupil has been accustomed to consider it in Arithmetic, and as he will contemplate it in this volume, is *Discontinuous Number.* Thus 5 grows till it becomes 9, by taking on additions of units of some conceivable value; as when we consider it as passing thus, 5, 5+1 or 6, 6+1 or 7, 7+1 or 8, 8+1 or 9. If the increment were any fraction, however small, *the form of the conception would be the same.*

As to *Continuous Number*, this is not the place for a full consideration of the idea; hence only a single illustration will be given. Time affords one. We usually conceive time as *discontinuous number*, as when we think of it as made up of hours, days, weeks, etc. But it is easy to see that such is not the way in which time actually grows. A period of one day does not grow to be a period of one week by taking on a whole day at a time, or a whole hour. or even a whole second. It grows by imperceptible increments (additions). These inconceivably small parts, by which time is actually made up, we call infinitesimals; and number, when conceived as made up of such infinitesimals, we call *Continuous Number.*

9. There are ***Three Branches*** of the *Science of Number*, viz., Arithmetic, Algebra, and the Calculus.

10. *Arithmetic* is the elementary branch of the Science of Number.

A more complete definition of Arithmetic is as follows:

11. *Arithmetic* treats of ***Discontinuous Number***—of its nature and properties, of the various methods

of combining and resolving it, and of its application to practical affairs.*

The leading topics of Arithmetic are:

1. Notation; *i. e.*, methods of representing number, as by the Arabic characters, 1, 2, 3, 4, etc., or by letters, as a, b, m, n, x, y, etc.

2. Properties of Numbers or deductions from the methods of Notation.

3. Reduction, as from one scale to another, from one denomination to another, from one fractional form to another, or, in short, from any one form of expression to another equivalent form.

4. The various methods of combining number, as by addition, multiplication, and involution.

5. Resolving number, as by subtraction, division, and evolution.

And all the above processes as effected by the use of any notation, and upon integral or fractional discontinuous numbers of any kind.

12. *A Proposition* is a statement of something to be considered or done.

ILLUSTRATION.—Thus, the common statement, "Life is short," is a proposition; so, also, we make, or state a proposition, when we say, "Let us seek earnestly after truth."—"The product of the divisor and quotient, plus the remainder, equals the dividend," and the requirement, "To reduce a fraction to its lowest terms," are examples of Arithmetical propositions.

13. *A Theorem* is a proposition which states a real or supposed fact, whose truth or falsity we are to determine by reasoning.

ILLUSTRATION.—"If the same quantity be added to both numerator and denominator of a proper fraction, the value of the fraction will be increased," is a *theorem*. It is a statement the truth or falsity of which we are to determine by a course of reasoning.

* It is obviously incorrect to say that "Arithmetic *is the Science of Number*." Arithmetic is no more the Science of Number than is Algebra, or the Calculus; the subject-matter of all these is number. Arithmetic is but a very small part of the Science of Number.

The word *Proposition* is frequently used as a synonym for *Theorem*; in fact, this is its most common use.

14. *A Demonstration* is the course of reasoning by means of which the truth or falsity of a theorem is made to appear.

This term is also applied to a logical statement of the reasons for the processes of a rule. A solution tells *how* a thing is done: a demonstration tells *why* it is so done. A demonstration is often called *proof*.

15. *An Axiom* is a proposition which states a principle that is so simple, elementary and evident, as to require no proof.

ILLUSTRATION.—Thus, "A part of a thing is less than the whole of it," "Equimultiples of equals are equal," are examples of axioms. If any one does not admit the truth of axioms, when he understands the terms used, we say that his mind is not sound, and that we cannot reason with him.

16. *A Problem* is a proposition to do some specified thing, and is stated with reference to developing the method of doing it.

ILLUSTRATION.—A problem is often stated as an incomplete sentence, as, "To reduce fractions to a common denominator."

17. *A Rule* is a formal statement of the method of solving a general problem, and is designed for practical application in solving special examples of the same class. Of course a rule requires a demonstration.

18. *A Solution* is the process of performing a problem or an example. It should usually be accompanied by a demonstration of the process.

19. SYNOPSIS.

- SCIENCE OF NUMBER.
  - ARITHMETIC.
    - DEFINITION.
    - TOPICS OF PURE ARITHMETIC.
      - Notation.
        - *With Arabic figures.*
        - *Roman.*
        - *Literal.*
      - Reduction.
        - *Of Scales.*
        - *Of Compound Nos.*
        - *Of Fractions.*
      - Combination.
        - *Addition.*
        - *Multiplication.*
        - *Involution.*
      - Resolution.
        - *Subtraction.*
        - *Division.*
        - *Evolution.*
      - Properties of Numbers.
        - *Divisibility.*
        - *Common Divisors.*
        - *Common Multiples.*
    - APPLICATIONS.
      - Numerous and varied, as
      - In barter, or common trade and sale;
      - In business operations based on Percentage;
      - In measurements; etc.
  - ALGEBRA.
  - CALCULUS.

[This skeleton will give an idea of the philosophical view of Arithmetic on which the arrangement in this book is based.]

CHAPTER I.

NOTATION.

SECTION I.

THE ARABIC NOTATION.

20. *A System of Notation* is a system of symbols by means of which quantities, the relations between them, and the operations to be performed upon them, can be more concisely expressed than by the use of words.

Numeration is usually defined as the art of reading numbers; but the word is never so used. In fact, this term is very little used, and when employed signifies the naming of the orders represented in a particular number, in regular succession, for the purpose of reading it.

21. In mathematics, as now studied, ***two sets of symbols*** are used to represent number, or quantity, viz., the *Arabic Symbols,* 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, called figures, and the *Common Letters,* a, b, c, d, . . . x, y, z.

The Roman method, by means of the seven capital letters, I, V, X, L, C, D, M, is not now used for computing, but only for marking the number of a chapter, section, or page of a book, or for some similar purpose.

22. *A Unit* is one.

An important characteristic of the Arabic system is that the *name* of the character and the *number of units it represents* are the same. Simple and natural as this feature seems, it is a singular fact that no other scheme for representing number, nor for representing vocal sounds (no alphabet) has it.

23. The character 0 is called *Zero,* or *Cipher,* and the other nine are called *Digits.* Their respective names and significations are *one, two, three, four, five, six, seven, eight, nine.*

24. ***The Decimal System of Notation*** is a system of grouping numbers into *tens,* and representing the number of groups by a digit, and the character of the group by the place of the digit.

Thus 10 units are grouped into 1 ten, 10 tens into 1 hundred, 10 hundreds into 1 thousand, etc.; and whether 5 represents 5 (groups of) tens, or 5 (groups of) thousands, depends upon its place with reference to the other figures used.

Some such system of grouping is a necessity, for, otherwise, we must needs have a character for each separate number.

25. These groups are called ***Orders,*** and a group of three orders is called a ***Period.*** The primary orders are *Units, Tens, Hundreds.* The periods are named *Units, Thousands, Millions, Billions, Trillions, Quadrillions, Quintillions, Sextillions, Septillions, Octillions, Nonillions, Decillions, Undecillions, Duodecillions,* etc.

This method of making the periods consist of three figures each is peculiar to the French and Americans. The English and other European nations make six-figure periods. Thus in England and Germany a billion is a million million, a trillion a million billion, etc.*

* In pointing off for convenience of reading, these nations place the commas just as we do, but read two of the groups thus formed as a period. Thus the six-figure method of reading makes 7,685,432,702,643,752 to be read 7,685 billion, 432,702 million, 643,752; whereas we read it, 7 quadrillion, 685 trillion, 432 billion, 702 million, 643 thousand, 752. The statement so frequently found in our arithmetics, that the French method prevails throughout the continent of Europe, is certainly a mistake; it is in common use only in France and this country, and has come into use here within two generations.

26. The ***Radix*** is the number which it takes of one order to make one of the next higher; thus the radix of the common system of notation is 10.

27. The ***Scale*** is the law of relation between the successive orders.

The word *scale* as thus used is from the Latin *scala*, a ladder, and has reference to the succession of steps by which we ascend from one order to another; thus from units to tens is one step, from tens to hundreds another, etc. Notice that these steps are *uniform*, each one being 10. In Compound Numbers the scale is usually irregular; thus from farthings to pence is 4, from pence to shillings 12, from shillings to pounds 20, the *denominations* in such cases corresponding to *orders* as above defined.

28. The ***Laws of the Arabic System*** of Notation are:

1. That each digit, in itself, always represents the same number of units.

2. That the order of these units (or of the digit) depends upon the place the digit occupies, reckoning to the left or right from units order.

3. That the sum of the values thus represented is indicated by any succession of figures.

4. That the 0 has no value, in itself, but is used to mark vacant orders.

[The purpose of this treatise does not require that the details of reading and writing numbers in the Decimal System be given.]

29. We can readily construct a system with any radix, as a *Binary* (radix 2), a *Ternary* (radix 3), *Quaternary* (radix 4), *Quinary* (radix 5), *Senary* (radix 6), *Septenary* (radix 7), *Octenary* (radix 8), *Nonary* (radix 9), *Undenary* (radix 11), *Duodenary* (radix 12), etc.

ILLUSTRATIONS.—In the *Quinary* system the 2d order to the left, reckoning from units, would be 5's instead of 10's, as in the decimal system; the 3d order would be 5's of 5's, or 25's, as in the decimal system it is 10's of 10's, or 100's; the 4th order would be 125's, the 5th 625's, etc. In like manner, in the *Septenary* system the 2d order would be 7's, the 3d 49's, the 4th 343's, the 5th 2401's, etc. In the *Duodenary* system the second order would be 12's, the 3d 144's, the 4th 1728's etc.

30. In any system the number of characters needed, including 0, is the same as the radix.

Thus in the Senary system we should have 0, 1, 2, 3, 4, 5, and the next number would be 10, which would be what we call 6. For the Undenary one new character, and for the Duodenary two new characters would be needed. Letting the characters in the Duodenary system be 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, θ, ψ, we should have the following:

In the *Duodenary*, θ, ψ, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 1θ,* corresponding to
In the Decimal, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22.

[For further information on this subject, see next chapter.]

31. Simple Numbers are numbers written in accordance with a uniform scale, and with a number of characters equal to the radix; *i. e.*, in accordance with the laws of the Arabic system, and with a uniform radix.

Such a number as 257, in any scale, is a simple number. But $4° 5' 2''$ is not a simple number, although the radix is uniform; there are not 60 characters. So also what are called *Duodecimals*, as 12 ft. $7'$ $10''$ $11'''$,† are not simple numbers, since there are not

* There is no established method of reading numbers written by other scales than the decimal, nor is there need of any, since such notation is merely speculative. A number represented thus, 234, in the Quinary system may be read, "2 25's, 3 5's, and 4." $78\theta602$ in the Undenary system may be read, "7 161051's, 8 14641's, θ (*theta*, used for 10) 1331's, 6 121's, and 1."

† Read "12 feet, 7 primes (or inches), 10 seconds, and 11 thirds." A prime is $\frac{1}{12}$ of a foot, a second $\frac{1}{12}$ of a prime, and a third $\frac{1}{12}$ of a second, etc. Hence the radix is 12.

12 characters. We can write this number as a simple number by adopting two new characters, as θ for 10, and ψ for 11. (See Article ***30.***) Thus the compound number 12 ft. 7′ 10″ 11‴ becomes the simple number 1070ψ in the *Duodenary* scale.

32. Compound Numbers are numbers written in an irregular scale.

The *Orders* in compound numbers are called *Denominations*, and hence we have the following definition :

33. A Compound Number consists of several related denominations written together, and to be read as one number.

For the Scales of Compound Numbers, we have the Tables of Compound Numbers at the close of the volume.

SECTION II.

*THE LITERAL NOTATION.**

34. In other departments of mathematics than Arithmetic, numbers or quantities are most frequently represented by the common letters of the alphabet, a, b, c, . . . m, n, . . . x, y, z.

35. The Primary Laws of the *Literal Notation* are :

1. *Any letter may be used to represent any number, provided it always means the same number in the same exercise or problem.*

* This is not the place for a full exposition of this notation, yet in order to completeness some account of it is necessary, and its great importance as pre-eminently the mathematical notation, demands that the pupil be made acquainted with it at the earliest practicable moment.

2. *When letters representing numbers are written side by side, as in a word, their product is indicated.*

3. *A number represented by Arabic characters written in connection with letters is subject to the same law as the letters, i. e., it is to be taken as one of the factors making up the entire number.*

ILLUSTRATIONS.—Thus a may represent any number, and b any other number, and if we write them thus, ab, the meaning is that the number represented by a is multiplied by the number represented by b. If a is used for 6, and b for 15, ab is 6×15, or 90. Again, if a stands for 12 and b for 432, ab is 12×432, or 5184. In like manner, if x stands for 7, $3x$ is 21. If $a = 4$, $b = 13$, and $x = 14$, $15abx = 15 \times 4 \times 13 \times 14 = 10920$.

Ex. 1. What is the value of $5am$, if $a = 3$, and $m = 7$? What if $a = 100$, and $m = 8$?

2. What is the value of $32bcy$, if $c = 5$, $b = 50$, and $y = 121$? If $c = 211$, $b = 27$, and $y = 1$?

3. What is the value of xy, if $x = 11$, and $y = 125$? If $x = 1$, and $y = 1$? If $x = 10$, and $y = 10$? If $x = 3$, $y = 3$?

4. If I buy a yards of cloth for b dollars per yard, what is the cost of the whole? *Ans.*, ab dollars.

Such an answer may seem to the student no answer at all, since it simply says that the cost of all is "a multiplied by b" dollars; but the advantage of such a notation will be seen as we progress. It is true that such an answer is just like answering the question, "If I buy 7 yards of cloth, at $3 per yard, what is the cost of all?" by saying, "7 times 3."

5. If a man buys m loads of potatoes containing x bushels each, on each of the 6 secular days of a week, how many bushels does he buy in all? *Ans.* $6mx$ bushels.

6. If a boys earn b cents each for each of x days, how much do they earn in all?

7. If a man draws a loads of bricks containing m bricks in a load, on each of 26 days, how many bricks does he draw in all.

36. An Expression like $7ax$, without any other joined with it by the signs + or —, is called a ***Term***, or a ***Monomial.***

If there are *two* such terms joined together by either of the signs + or —, the two taken together are called a ***Binomial***, as $6bx + 2ay$, or $10x - 3ay$.

If three terms are joined in this way it is called a ***Trinomial***, as $3ay - 2ab + 21x$.

Any expression consisting of more than one term is, in general, called a ***Polynomial.***

Ex. Point out the monomials, binomials, trinomials, and polynomials, in the following: $2ax - 3b$, $5xy - 6cd + a - 2y$, $3amxy$, $c - d$, $a + m$, $a + b + c - d$, $225abcd$, $abcd$, $a - b$, ab, $c - xy + ax$, $x + y$, $10a + 3xy$.

SECTION III.

OF UNITS, INTEGERS, AND FRACTIONS.

37. In the Arabic Notation some one of the orders, or denominations, is always assumed as the ***Primary Unit***, from which higher orders or denominations are written to the left, and lower ones to the right.

38. The Primary *Unit* of *Simple Numbers* is 1 of the units order; and the higher orders are tens, hundreds, thousands, ten-thousands, hundred-thousands, etc. The

lower orders are tenths, hundredths, thousandths, ten-thousandths, hundred-thousandths, millionths, ten-millionths, etc.

39. When higher and lower orders of simple numbers are written together, a period (.) is written at the right of units order to indicate its place. This period is called the *Decimal Point.*

HIGHER ORDERS.								LOWER ORDERS.							
Millions.	Hund. Th.	Ten Th.	Thousands.	Hundreds.	Tens.	*Prim. Units.*	Dec. Pt.	Tenths.	H'ndredths.	Th'sandths.	T.-Th'ths.	H.-Th'ths.	Millionths.	T.-Mill'ths.	H.-Mill'ths.
9	4	5	6	2	3	7	.	8	1	4	5	6	8	3	7

40. In our common *Compound Numbers* the following are units. They are called *Standard* Units because established by law:

1. In Measures of Extension, the *Yard.*
2. In Liquid Measure, the *Wine Gallon.*
3. In Dry Measure, the *Winchester Bushel.*
4. In Weight, the *Troy Pound.*
5. The *Unit of Time* is the *Mean Solar Day.**
6. In United States Money, the *Dollar.*
7. In French Money, the *Franc.*
8. In German Money, the *Mark.*

* The Sidereal Day, *i. e.*, the time elapsing between two successive passages of the meridian by one of the fixed stars, is the natural unit of time, as this is unvarying. The time between two successive passages of the *sun* across the meridian varies through the year, being greatest in January and least in July. The *mean solar* day is the average of these actual solar days.

The *Yard* (36 *in.*), as is said above, is the standard unit of measures of extension, as of Long, Square, and Cubic Measures, in this country, Canada, and England. The present standard yard kept by our government at Washington is a brass rod, 82 *in.* long, obtained from England, and is to be used for comparison at the temperature of 62° Fah. Being made by an instrument maker, Troughton, it is sometimes called Troughton's Scale.

The history of this measure in England is in brief this: In 1742, Mr. George Graham made experiments to ascertain the exact length of a pendulum beating seconds at London. Calling this length, as he determined it (now known to be incorrect) 39.13 inches, he made scales, which were copied with very great accuracy by Mr. Bird, an instrument maker. A brass bar with two gold studs placed 36 inches apart, made by Mr. Bird and labeled "Standard Yard, 1760," was kept in the Parliament House as the standard until the House was burned in 1834. This was restored from copies in the possession of the Astronomical Society in comparison with some others, a learned committee having reported that it would be better to restore it thus than to attempt its restoration on philosophical principles. One of the four restorations thus made was furnished to our government.

A *Liquid Gallon* is 231 cubic inches.*

The *Winchester Bushel* is 2150.4 *cu. in.*, very nearly.

A *Troy Pound* is 5760 grains, a liquid gallon containing 58372.1754 grains of distilled water at its maximum density (39.83° Fah.), weighed in the air, the barometer being at 30 *in.*

The *Avoirdupois Pound* is 7000 grains.

The *Gold Dollar* (.9 pure gold, and .1 alloy of silver and copper) weighs 25.8 grains.

Thus we see that all our weights and measures are really based on the linear yard.†

41. In the ***Metric System*** of weights and measures, the *Unit of Extension* is the *Meter* (39.37 *in.*); of *Meas-*

* The English Imperial Gallon is larger than ours, being 277.274 *cu. in.*

† The length of the seconds-pendulum being, *theoretically*, the basis of the linear unit, the revolution of the earth on its axis is, *theoretically*, the basis of all our measures. But, as the above statement of facts shows, this is only theoretical. Actually, the length of the yard has no more dependence on the length of the seconds-pendulum than if the device had never been thought of.

ures of Capacity, the *Liter* (1.0567 liquid quarts, or .908 dry quarts); and of *Weight,* the *Gram* (15.432 grains).

The French government undertook very elaborate measures for ascertaining the exact length of a quadrant of a meridian (*i. e.*, a quarter of the circumference of the globe), designing that the Meter should be exactly $\frac{1}{10000000}$ part of this quadrant. This, however, is now known to have been inaccurately measured, so that the Meter is no more a natural measure than the yard—both are arbitrary. The Meter in use is a little shorter than it was designed to be. (See Appendix.)

All the other units of the Metric System are deduced from the Meter. (See Appendix.)

42. *An Integer* is an entire unit, or collection of entire units; *i. e.,* it is a *whole number* in distinction from a fraction.

43. *A Fraction* is a number representing one or more of the equal parts into which a unit, or some number taken as a whole, is conceived to be divided.

44. *A Common Fraction* is a fraction which arises from conceiving the unit, or number, divided into any number of equal parts (other than a number the same as the radix), as convenience may dictate.

45. *A Decimal Fraction* is a fraction in the decimal system which arises from dividing the unit by 10, and these 10ths again by 10, etc., *i. e.,* from a decimal division of the unit.

46. *The Denominator* of a fraction is the number which indicates into how many equal parts the unit, or number, is conceived to be divided. ***The Numerator*** is the number which indicates how many of these equal parts are represented by the fraction.

47. A Common Fraction is written in figures by writing the numerator above the denominator with a line between them.

For the principle of the method of writing a *Decimal Fraction* see (*39*). Only the numerator of such a fraction is usually written, the denominator being readily inferred.

[The following practical directions for reading and writing Decimals are too important to be omitted; few students are found who are expert in these simple processes. If exercises are needed other than can be extemporized, the author's TEACHER'S HAND-BOOK will supply them in abundance.]

48. To Read a Decimal.—*I. Numerate the fraction; that is, begin at the decimal point and name the orders to the right, and bear in mind the name of the lowest, or right-hand order.*

II. Read the expression just as a whole number, and then pronounce the name of the lowest or right-hand order.

49. To Write a Decimal.—*Write the numerator as a whole number. Then beginning at the* RIGHT, *apply the decimal numeration, calling the right-hand figure tenths, the next at the left hundredths, etc., filling all vacant orders with 0's, till the name of the order designated by the denominator is reached; at the left of this write the decimal point.*

SECTION IV.

SYMBOLS OF OPERATION.

50. The sign +, called *plus*, indicates addition.

51. The sign —, called *minus*, indicates subtraction, the number preceding it being the minuend, and the number following, the subtrahend.

52. The sign $\times$ indicates multiplication. Multiplication is also indicated by the period placed in the middle of the line, thus $4 \cdot 5$, and in the literal notation by writing letters in succession without any sign between them, as abc, which means the same as $a \times b \times c$. (See ***35.***)

53. The sign $\div$, or :, indicates division, the number before the sign being the dividend, and the number after the sign the divisor. Division is also indicated by writing the dividend above the divisor with a line between them $\left(\frac{16}{4}, \frac{a}{b}\right)$*, by writing the divisor on the right of the dividend with a curved line between [4)16, $a)b$], or by writing the divisor in a similar manner at the left.

54. The sign $=$, called the *Sign of Equality*, signifies that the expressions between which it is placed are equal.

55. The signs ( ), [ ], { }, and a horizontal line over a number are *Symbols of Aggregation*, and signify that the expression enclosed is to be taken as a whole; thus $(3 + 6)(2 + 5)$ means that $3 + 6$, or 9, is to be multiplied by $2 + 5$, or 7, so that $(3 + 6)(2 + 5) = 63$, while $3 + 6 \times 2 + 5 = 3 + 12 + 5 = 20$. $5(6 - 3) = 5 \times 3 = 15$, while $5 \times 6 - 3 = 27$. $\overline{44 - 2} \div \overline{10 - 3} = 6$, while $44 - 2 \div 10 - 3 = 41 - \frac{1}{5} = 40\frac{4}{5}$.

56. The colon, :, written between numbers indicates the ratio of the former to the latter, which is the same thing as the former divided by the latter; thus $8 : 4$ may be read "the ratio of 8 to 4," or "8 divided by 4," as they

* Read "a divided by b," the same as $a \div b$. The reading "a over b" is both inexpressive and inelegant.

are equivalent expressions. The equality of two ratios is indicated by the double colon, ::, as in proportion. This sign is exactly equivalent to =.

57. The sign $\sqrt{\ }$, called the *Radical Sign*, indicates the square root of the number over which it is placed, that is, one of the two equal factors. $\sqrt[3]{\ }$ indicates the cube root, that is, one of the three equal factors, etc.

[For a fuller analysis of this subject, see the author's COMPLETE SCHOOL ALGEBRA. Other symbols will be explained as needed.]

SYNOPSIS.

- NOTATION.
  - DEFINITIONS.
  - SYMBOLS.
    - OF QUANTITY.
      - ARABIC.
        - What.
        - Names.
        - Significations.
          - *Simple Nos.*
            - Unit.
            - Order.
            - Period.
            - Radix. / Scale.
              - *Decimal.*
              - *Others.*
          - *Comp. Nos.*
            - Units—Standard.
            - Scales.
          - *Fractions.*
            - Common.
            - Decimal.
        - Laws of use.
      - LITERAL.
        - What.
        - Laws of.
    - OF OPERATION.
      - ADDITION.
      - SUBTRACTION.
      - MULTIPLICATION.
      - DIVISION.
      - RATIO.
      - EQUALITY.
        - =
        - ::
      - AGGREGATION.
      - ROOTS.

CHAPTER II.

REDUCTION.

SECTION I.

FROM ONE SCALE TO ANOTHER.

58. *Reduction* is changing the form of an expres sion without altering its value.

It will be seen that, philosophically, to reduce an expression from one form to another is but to change the notation by means of which the number is represented. Thus having 277 in the decimal scale, or notation, to reduce it to 2102 in the quinary scale, is but to change the notation. So also given £3 4*s.* 10*d.* to reduce it to 778*d.*, is simply to change the notation. Again $\frac{15}{18}$ and $\frac{5}{6}$, $\frac{17}{2}$ and $8\frac{1}{2}$, $\frac{3}{8}$ and .375, are severally but different notations for the same number. The processes of Reduction are, therefore, immediate consequences of notation. But, in an elementary course, reduction cannot well be treated till after the fundamental rules; since the operations of addition, multiplication, etc., are exceedingly convenient, if not absolutely necessary in effecting the change of notation.

Ex. 1. According to the laws of the Arabic notation, what does 324 in the Quinary System signify in the Decimal System?

Solution.—The 4 represents 4 simple units, the 2 represents

2 *fives*, or 10, and the 3 represents 3 *fives* of *fives*, or 3 *twenty-fives*, *i. e.*, 75. Hence $324_5{}^* = 75 + 10 + 4 = 89_{10}{}^*$.

2. Show that $1302_5 = 202_{10}$. $10200_5 = 675_{10}$.
3. Show that $40210_5 = 2555_{10}$. $1111_5 = 156_{10}$.
4. Show that $10_5 = 5_{10}$. $100_5 = 25_{10}$. $1000_5 = 125_{10}$.
5. Reduce 758_{10} to the quinary scale.

SOLUTION.—In 758 there are 151 *fives*, and 3 units over. In 151 *fives* there are 30 *fives of fives*, or 25's, and 1 *five* over. In 30 25's there are 6 125's and 0 25's over, and in 6 125's there is 1 625 and 1 75 over. Hence $758_{10} = 11013_5$.

OPERATION.

$$\begin{array}{r|l} 5\,)\,758 & -\ 3 \\ 5\,)\,151 & -\ 1 \\ 5\,)\,30 & -\ 0 \\ 5\,)\,6 & -\ 1 \\ 1 & \end{array}$$

6. Reduce the following to the quinary scale: 68_{10}, 111_{10}, 20_{10}, 10_{10}, 3802_{10}, 100_{10}.

Results. 233_5, 421_5, 40_5, 20_5, 110202_5, 400_5.

7. Solve the following both by reducing the number from the unusual to the decimal scale and *vice versa*, thus making two exercises of each:

$$6432_{10} = 24516_7. \qquad 201_4 = 33_{10}.$$
$$4503_{10} = 32503_6. \qquad 2343_{10} = 210213_4.$$
$$800_{10} = 1078_9. \qquad 1000_3 = 27_{10}.$$

8. Letting θ and ψ be the next two consecutive digits after 9, reduce $3\psi 0\theta 2_{12}$ to the decimal scale. *Ans.* 81338.

It will be observed that it is necessary to have a number of characters, including 0, equal to the radix used, and no more; thus, if the radix is 5 we must have 5 characters. We need no character 5, since 10_5 is 5.

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* We shall indicate the scale by writing the radix as a subscript. For method of reading, see foot-note, page 10.

9. Show that $6543_7 = 3173_9$.

SOLUTION.—In 6543_7 there are 520_7 *nines* and 3 units over. In 520_7 *nines* there are 40_7 81's and 7 *nines* over. In 40_7 81's there are 3 729's and 1 81 over. Hence $6543_7 = 3173_9$.

OPERATION.

$$\begin{array}{r@{\,)\,}l}
{}^*9 & \underline{6543_7} \;—\; 3 \\
9 & \underline{520} \;—\; 7 \\
9 & \underline{40} \;—\; 1 \\
 & 3
\end{array}$$

10. Show that $1000_6 = 3120_4 = 1331_5$.

59. *Let the student write rules for reducing from the decimal scale to any other, and vice versa.*

Every rule should be accompanied with a *Demonstration*, *i. e.*, an orderly statement of the reasons for each step in the process.

60. Generalization.—There is but a single principle running through all arithmetical reductions; viz., *To pass from a higher denomination, or order, to a lower, multiply by the number which it takes of the lower to make one of the higher. To pass from a lower to a higher, divide by this number.*

11. Show in accordance with the above principle that $3458_{10} = 24002_6$.

SOLUTION.—Since 3458 may be understood as 3458 units, we state the problem, "To reduce 3458 *units* to sixes, thirty-sixes, two hundred-sixteens, etc." Now as 6 units make 1 six we divide by 6; or, there will be $\frac{1}{6}$ as many sixes as units, $\frac{1}{6}$ as many 36's as 6's, etc. This is, therefore, the ordinary case of *Reduction Ascending*.

OPERATION.

$$\begin{array}{r@{\,)\,}l}
6 & \underline{3458} \;—\; 2 \\
6 & \underline{576} \;—\; 0 \\
6 & \underline{96} \;—\; 0 \\
6 & \underline{16} \;—\; 4 \\
 & 2
\end{array}$$

* The principle on which this division is performed is the one in ordinary use; but it must be borne in mind that 6 of the 4th order makes 42 of the 3d, which with the 5 makes 47, etc.

12. Reduce 24002_6 to the decimal scale, explaining as reduction descending.

OPERATION.

1296's.	216's.	36's.	6's.	Units.
2	4	0	0	2

6	
16	216's.
6	
96	36's.
6	
576	6's.
6	
3458	units.

SOLUTION.—This is simply a case in reduction descending. Thus 2 1296's make 6 times as many 216's, and adding in the 4 216's, we have 16 216's. 16 216's make 6 times as many 36's, or 96 36's, etc.

61. Review the preceding examples in this section, giving similar solutions.

SECTION II.

REDUCTION OF FRACTIONS.

62. It is the purpose of this section to show how all the operations known as *Reductions of Fractions* conform to the common principle of reduction as stated in (***60***).

1. Explain the reduction of $\frac{4}{8}$ to $\frac{2}{4}$ in accordance with (***60***).

SOLUTION.—Since 2 8ths make 1 4th, in 4 8ths there are $\frac{1}{2}$ as many 4ths as 8ths, *i. e.*, 4÷2. or 2 4ths, written $\frac{2}{4}$.

It will be observed that in the ordinary process of reducing a fraction to lower terms, the number by which we divide is simply the number which it takes of the given denomination to make 1 of that denomination to which we reduce the fraction by the division. The reduction proper is performed by dividing the numerator. The division of the denominator simply obtains the number which indicates to what denomination we have reduced the fraction.

2. Explain as above that $\frac{50}{75} = \frac{10}{15} = \frac{2}{3}$.

How many 75ths in a 15th? How many 15th's make a 3d?

3. Explain as above $\frac{315}{420} = \frac{45}{60} = \frac{9}{12} = \frac{3}{4}$.

It will be seen that this is the common case of *Reduction Ascending*.

4. Reduce $\frac{3}{7}$ to 35ths.

In 1 7th how many 35ths? Then how many 35ths in 3 7ths? Is this reduction ascending or reduction descending?

5. Reduce $\frac{5}{11}$ to 88ths. $\frac{4}{13}$ to 65ths.

6. Reduce $\frac{56}{7}$ to integers, *i. e.*, to units.

How many 7ths make 1 unit? Then how many units in 56 7ths? Is this reduction ascending or reduction descending?

7. Reduce $\frac{236}{4}$, $\frac{348}{6}$, $\frac{1331}{11}$, and $\frac{5865}{5}$ to integers.

8. Reduce $\frac{11}{4}$ to integers.

Since there are 4 4ths in 1 unit, in 11 4ths there are as many units as 4 is contained times in 11, *i. e.*, 11÷4, which is 2 and 3 4ths remaining, written $2\frac{3}{4}$.

9. Reduce 7 to thirds.

Is this effected by multiplication or by division? Is it reduction ascending or reduction descending?

10. Reduce $5\frac{2}{7}$ to 7ths.

The identity of the two processes in the margin is apparent. The rationale of the two processes is also exactly the same.

s.	*d.*	Units.	Sevenths.
5	2	5	2
12		7	
62*d.*		37 sevenths.	

11. Reduce $12\frac{2}{5}$, $101\frac{3}{7}$, and $57\frac{11}{13}$ to improper fractions, explaining as above.

12. Show that $\frac{4}{7}$ and $\frac{3}{4}$ are respectively $\frac{16}{28}$ and $\frac{21}{28}$, explaining as above.

How many 28ths in 1 7th? How many in 1 4th?

13. Explain as above that $\frac{1}{2}$, $\frac{7}{8}$, and $\frac{7}{12}$ are respectively $\frac{12}{24}$, $\frac{21}{24}$, and $\frac{14}{24}$.

Thus the reduction of fractions to forms having common denominators is seen to be but the ordinary case of reduction descending. The method of determining to what common denomination the several fractions may be reduced, *i. e.*, of finding the common denominator, is not within our present purpose; in fact there is an infinite number in each case.

14. Explain as above the reduction of $2\frac{2}{3}$, 3, $\frac{3}{4}$, and $1\frac{5}{6}$ to 12ths. To 24ths. To 48ths.

---

15. Reduce $\frac{3}{5}$ to a decimal fraction.

SOLUTION.—Since there are 2 10ths in 1 5th, in 3 5ths there are 2 times 3 10ths, or 6 10ths, written as a decimal .6.

Or, since in any number of units there are 10 times as many tenths, in $\frac{3}{5}$ of a unit there are 10 times $\frac{3}{5}$, or $\frac{30}{5}$ 10ths, *i. e.*, 6 10ths, written as a decimal .6.

16. Reduce $\frac{7}{8}$ to a decimal, showing that it is but a case of reduction descending.

How many 10ths in $\frac{7}{8}$ of a unit? *Ans.*, 8, and $\frac{6}{8}$ of a 10th. How many 100ths in $\frac{6}{8}$ of a tenth? *Ans.*, 7 hundredths and $\frac{4}{8}$ of a hundredth. In $\frac{4}{8}$ of a hundredth there are 5 thousandths. Hence $\frac{7}{8}$ = .875.

Or, how many thousandths in 1 8th? *Ans.*, 125 thousandths. Then in 7 8ths there are 125 times 7 thousandths, or .875.

Or, again, since in 1 unit there 1000 thousandths, in $\frac{7}{8}$ of 1 unit there are 1000 times $\frac{7}{8}$, or $\frac{7000}{8}$ thousandths, *i. e.*, .875.

17. Reduce $\frac{3}{4}$, $\frac{7}{20}$, $\frac{11}{40}$, $\frac{4}{25}$, and $\frac{3}{50}$ to decimals, explaining the process as above.

18. Reduce $\frac{2}{3}$, $\frac{5}{6}$, $\frac{4}{7}$, $\frac{6}{11}$, and $\frac{3}{13}$ to decimals, explaining as above.

$\frac{6}{11}$ are equivalent to 10 times $\frac{6}{11}$ tenths, or $\frac{60}{11}$ tenths = $.5\frac{5}{11}$. So $\frac{5}{11}$ tenths are equivalent to 10 times $\frac{5}{11}$ hundredths, or $\frac{50}{11}$ hundredths = $4\frac{6}{11}$ hundredths. Hence $\frac{6}{11} = .54\frac{6}{11}$, and the process can be extended as far as we please.

63. *A Repetend* is a decimal fraction which, after a certain order is reached, consists of a figure, or a set of figures in a given order, continually repeated. The set of figures thus repeated constitutes a *Period*. When the period commences with tenths the decimal is a *Pure Repetend*; when with any lower order, a *Mixed Repetend*.*

64. *Prop.*—*A common fraction can always be expressed exactly as a decimal or by means of a repetend.*

DEM.—As every remainder must be at least 1 less than the divisor, there can be only as many *different* remainders, after we begin to annex 0's in the reduction, as there are units in the entire divisor. Now, if one of these remainders is 0, the work terminates at that point. But, if no remainder is 0, there must needs recur one which we have had before, either before or next after we have reached the possible limit of different remainders. Now when a remainder occurs which we have had before, the figures obtained in the quotient between these two will necessarily recur in the same order.

65. How many figures is it possible to have in any repetend; *i. e.*, how does this number of figures compare with the denominator of the fraction from which the repetend arises?

19. To what repetends do the following give rise: $\frac{4}{81}$, $\frac{2}{33}$, $\frac{7}{51}$, $\frac{1}{7}$, $\frac{3}{7}$, $\frac{12}{17}$?

How does the number of figures in a period compare with the denominator in each of the last three?

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20. Reduce .375 to a common fraction.

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* This simple terminology is suggested as one which is actually in use, and which avoids multiplying terms on so unimportant a subject. To say nothing of the inelegance of many of the terms proposed by some writers, there is no unanimity in their use, and no necessity for them.

Solution.—Writing the denominator, 1000, of this fraction, and omitting the decimal point which is used to indicate the unexpressed denominator, we have $\frac{375}{1000}$. The reduction of $\frac{375}{1000}$ to $\frac{3}{8}$ is explained under Ex. 1, etc.

21. In like manner reduce .48, .035, .0075, .5, .8, .12, .4325, and .0625 to common fractions.

22. Reduce 1.5, 2.25, 8.455, 10.625, 7.5, 130.025, and 30.0375 to common fractions.

Since 2 is 200 *hundredths*, and .2 is 20 *hundredths*, 2.25 is 225 *hundredths;* or, written as a common fraction, $\frac{225}{100} = \frac{9}{4}$.

It is an important direct consequence of the Arabic Notation that *any number represented in the Arabic Notation may be read as so many of the lowest order as the figures represent when read together as a whole number.* This is shown in the solution above. Let the student show that it would be so in any scale, as the quinary, the duodenary, or any other.

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23. Reduce $.\dot{7}$ to a common fraction.*

Solution.—Observing that .111, etc., to infinity, or $.\dot{1}$, arises when we attempt to reduce $\frac{1}{9}$ to a decimal, and that .1111, etc., to infinity, or $.\dot{1}$, multiplied by 7 makes $.\dot{7}$, we find that $.\dot{7} = \frac{7}{9}$.

Or, more briefly, since $.\dot{1} = \frac{1}{9}$, $.\dot{7}$ which is 7 times $.\dot{1}$ equals 7 times $\frac{1}{9}$, or $\frac{7}{9}$.

24. Reduce $.\dot{2}$, $.\dot{3}$, $.\dot{4}$, $.\dot{5}$, $.\dot{6}$, $.\dot{8}$, $.\dot{9}$, to common fractions, explaining as above.

Query.—We readily produce $.\dot{8}$ from $\frac{8}{9}$ by the ordinary method of reducing a common fraction to a decimal. Can we produce $.\dot{9}$ from $\frac{9}{9}$, or 1, in a like manner?

* That this signifies repetend 7, the student is supposed to know from his study of Elementary Arithmetic. So also he should know that $5.4\dot{3}2\dot{6}$ means 5.4326326326, etc.

25. Reduce $.0\dot{1}$, $.0\dot{7}$, $.0\dot{8}$, $.0\dot{4}$, $.0\dot{5}$, and $.0\dot{9}$ to common fractions.

What common fraction when reduced to a decimal gives the repetend .01111, etc., or $.0\dot{1}$? $.0\dot{8}$, or .08888, etc., is how many times $.0\dot{1}$?

26. Reduce $.00\dot{1}$, $.00\dot{6}$, $.00\dot{9}$, $.0000\dot{7}$, $.000\dot{5}$, and $.000\dot{4}$ to common fractions.

27. Reduce $.\dot{0}\dot{1}$, $.\dot{0}\dot{2}$, $.\dot{0}\dot{6}$, $.\dot{0}0\dot{4}$, $.\dot{0}00\dot{5}$, $.\dot{0}0\dot{8}$, and $.\dot{0}000\dot{7}$ to common fractions.

Remember that $.\dot{0}0\dot{4} = .004004004$, etc. What common fraction when reduced to a decimal gives $.\dot{0}0\dot{1}$? What $.\dot{0}00\dot{1}$? $.\dot{0}0\dot{5} = \frac{5}{999}$.

28. Reduce $.\dot{2}3\dot{5}$, $.\dot{4}\dot{3}$, $.\dot{5}14\dot{2}$, $.\dot{8}\dot{1}$, $.\dot{7}\dot{5}$, $.\dot{4}8\dot{1}$, $.\dot{0}2\dot{7}$, and $.\dot{3}25\dot{6}$ to common fractions.*

How does $.\dot{4}8\dot{1}$ compare with $.\dot{0}0\dot{1}$? $.\dot{6}5\dot{7} = \frac{73}{111}$.

29. Reduce $.0\dot{2}\dot{7}$, $.00\dot{2}\dot{7}$, $.000\dot{2}\dot{7}$, $.0\dot{4}3\dot{5}$, $.00\dot{4}3\dot{5}$, $.000\dot{4}3\dot{5}$, $.00\dot{2}9\dot{8}$, and $.000\dot{8}4\dot{6}$ to common fractions. $.0\dot{4}3\dot{5} = \frac{29}{666}$.

30. Reduce $.32\dot{6}$, $.53\dot{1}\dot{2}$, $.6\dot{2}4\dot{5}$, $.301\dot{5}2\dot{6}$, $.3\dot{7}$, $.403\dot{6}41\dot{2}$, $.0028\dot{1}02\dot{1}$, and $.612\dot{4}$ to common fractions.

$.6\dot{2}4\dot{5} = .6 + .0\dot{2}4\dot{5}$. Now $.6 = \frac{6}{10}$, and $.0\dot{2}4\dot{5} = \frac{245}{9990}$. Hence $.6\dot{2}4\dot{5} = \frac{6}{10} + \frac{245}{9990} = \frac{6239}{9990}$.

66. *Let students now write out rules for reducing both varieties of Repetends to common fractions.*

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* The fractions obtained should be put in their lowest terms, *and the results verified* by reducing them back to decimals.

SECTION III.

REDUCTION OF DENOMINATE NUMBERS.

67. As in the preceding section we showed how the principle of (**60**) covers all cases of reduction of Fractions, so in this our main purpose is to show that all cases of reduction of Denominate numbers are covered by the same principle.

Ex. 1. Reduce $\frac{3}{4}$ *lb. Av.* to ounces.

Solution.—Since 16 *oz.* make 1 *lb.*, in any number of pounds there are 16 times as many ounces as pounds. Hence in $\frac{3}{4}$ *lb.* there are $\frac{3}{4} \times 16$, or 12 *oz.*

2. In 3 *lb. Av.* how many ounces?
3. In .75 *lb. Av.* how many ounces?

Same solution as above, merely changing the number representing the pounds.

4. In $3\frac{3}{4}$ *lb. Av.* how many ounces?

$3\frac{3}{4} \times 16 = 60$. Hence there are 60 ounces.

5. In 3.75 *lb. Av.* how many ounces?

$3.75 \times 16 = 60$. Hence there are 60 ounces.

6. Reduce £4 10*s.* 7*d.* to pence.
7. Reduce $\frac{3}{4}$*s.* to farthings.
8. Reduce $\frac{1}{96}$*s.* to farthings.

Notice that in any case the reduction from a higher to a lower denomination is effected simply by *multiplication*. If it is desired to change the form of notation from fractional to integral, from one form of fraction to another, the operation falls under some one of the processes shown in the preceding section to conform to the general principle of reduction.

9. Reduce 126720 *in.* to miles.

Solution.—Since 12 *in.* make 1 *ft.*, in any number of inches there are $\frac{1}{12}$ as many feet as inches. Hence in 126720 *in.* there are 126720÷12, or 10560 *ft.*, etc.

10. Reduce 8448 *in.* to miles.

8448 *in.* make 8448 ÷ 12, or 704 *ft.* 704 *ft.* make 704 ÷ 5280, or $\frac{704}{5280} = \frac{2}{15}$ *miles*. Reasoning exactly as above.

11. Reduce 12 *pwt.* 16 *gr.* to pounds.

$$\begin{array}{r|l} 24 & 16 \\ 20 & 12\tfrac{2}{3} = \tfrac{38}{3} \\ 12 & \tfrac{19}{30} \\ & \tfrac{19}{360} \end{array} \qquad \begin{array}{r|l} 24 & 16 \\ 20 & 12.666+ \\ 12 & .6333+ \\ & .052\dot{7} \end{array}$$

Give in either case the ordinary explanation for reducing from lower to higher denominations *by division*.

Perform the following, reversing the operation to prove the work, and giving the ordinary explanation:

12. 1 *mi.* to meters.
13. 20 *lb. Av.* to kilos.
14. 640 *acres* to ares.
15. 1.6 *inches* to feet.
16. 46850 *cu. ft.* to barrels.
17. 1 *mi.* to kilometers.
18. 1 *gal.* to liters.
19. 1 *stere* to cords.
20. 7 *mo.* 25 *da.* to years.
21. $\frac{1}{360}$ *yr.* to months.
22. $\frac{1}{15}$ *yr.* to days.
23. 10 *stere* to cords.
24. $\frac{3}{8}$ *pt.* to barrels.
25. .05 *ft.* to millimeters.
26. 34 *decameters* to yards.
27. 10 *bush.* to cu. inches.
28. 1 *rd.* to meters.
29. 1 *hectare* to acres.
30. 1 *hectoliter* to bushels.
31. 3 *qt.* 1 *pt.* to gallons.
32. $34.50 to French currency.
33. $81.45 to German currency.
34. 3000 *fr.* to United States currency.
35. 7 *cu. ft.* 112 *cu. in.* to cubic inches.

36. 42 *millimeters* to inches.
37. $2\frac{1}{2}$ *lb. Av.* to pounds Troy.
38. 642 *decimeters* to inches.
39. 486252 *sq. ft.* to acres.
40. .91225 *lb.* to integers of lower denominations Troy.
41. $5.62542 to integers of lower denominations.
42. $\frac{5}{12}$ of a mile to integers of lower denominations.
43. $\frac{7}{8}$ of a cord to cubic feet.
44. $\frac{2}{3}$ of a rod to miles.
45. $2\frac{3}{4}$ barrels to bushels.
46. 6 ounces Apothecaries' weight to Avoirdupois.
47. 25 milligrams to ounces Avoirdupois.
48. 24 *gr.* Apothecaries to grams.
49. 250 millimeters to inches.
50. $3^\circ\ 15'\ 20''$ to seconds.
51. $3^\circ\ 18'\ 42''$ to degrees and decimals of a degree.
52. $1000 to francs. To marks.
53. 68° Fahrenheit to Centigrade.
54. 22° Centigrade to Fahrenheit.
55. 98° Fahrenheit to Centigrade.
56. 72° Centigrade to Fahrenheit.

SYNOPSIS.

REDUCTION.
- WHAT.
- BUT ONE PRINCIPLE (*60*).
- FROM ONE SCALE TO ANOTHER.
- OF DENOM. NOS.
  - Ascending.
  - Descending.
  - *Including Integers, and Fractions, Com. and Dec.*
- OF FRACTIONS.
  - To lower or higher terms.
  - Improper to Whole or Mixed Numbers.
  - Whole or Mixed Numbers to Imp. Frac'ns.
  - To forms having Common Denominator:

CHAPTER III.

COMBINATION OF NUMBERS.

SECTION I.

ADDITION.

68. It is the leading purpose of this section to show that all the processes in Arithmetic which we call Addition are based upon the same fundamental principles.

69. ***Abstract Numbers*** are numbers to which no other signification is attached than that of mere number, as 5, 40, 275.

70. ***Concrete Numbers*** are numbers applied to some objects, or to which some other signification than that of mere number is attached,* as 5 *men*, 40 *dollars*, 275 *pounds*, etc.

71. ***The Order of Development*** of the subject of Addition is,

1. *By counting we ascertain the sum of the digits, two and two, i. e., make the Addition Table.*

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* *Positive* and *negative* numbers are in some sense concrete. See author's COMPLETE SCHOOL ALGEBRA (*54*), *et seq.* So also numbers which represent *time*, *geometrical concepts* (as lines, surfaces, and solids), *force*, and *motion*, are concrete.

2. *We commit to memory these sums, i. e., learn the Addition Table.*

3. *By means of a knowledge of the sums of the digits taken two and two we ascertain the sum of any number of numbers.*

72. *The Sum* of two or more numbers is the number which they make when united.*

73. *Addition* is the process of finding the sum of two or more numbers by means of a knowledge of the sums of the digits taken two and two.*

There are two ways of finding the sum of numbers, viz., by *counting* and by *adding*.

74. *The Fundamental Principles* of the processes that we call Addition are the *three* following

PROPOSITIONS.

1. *Only abstract numbers, or concrete numbers representing things of the same kind can be added together.*

2. *Only like orders or denominations can be directly added together.*

3. *In adding related orders or denominations it is practically most convenient to add the lowest orders or denominations first;* since, by so doing, we are enabled to

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* These definitions are scarcely broad enough to cover the case of addition of literal numbers; they apply more particularly to the Arabic Arithmetic. See author's COMPLETE SCHOOL ALGEBRA (*64*, *65*). A complete mathematical definition of addition of numbers must embrace the ideas of both sets of definitions. We do not call the sum of 8 and 7, 12 and 3, but fifteen (*i. e.*, 5 and 10), solely because the latter is expressed in the simplest terms consistent with the Arabic decimal notation, which the former is not.

determine whether the sum of any lower order or denomination makes any integers of the higher; and, if it does, we are enabled to carry this sum forward and unite it with those higher orders as we proceed.

Ex. 1. Find the sum of 53742, 38027, 6052, 845, 76, 4, 370, and 5603.

OPERATION.

53742
38027
6052
845
76
4
370
5603
104719

2. Find the sum of 10 *gal.* 3 *qt.* 1 *pt.*, 13 *gal.* 1 *pt.*, 3 *qt.* 1 *pt.*, 2 *qt.*, 1 *qt.* 1 *pt.*, 5 *gal.* 2 *qt.* 1 *pt.*

OPERATION.

10	3	1
13	0	1
	3	1
	2	0
	1	1
5	2	1
31	1	1

SOLUTION.*—In both cases we write the numbers so that like orders (or denominations) shall fall in the same column, in order that we may the more readily see how many there are of any one order (or denomination).

We begin to add with the lowest order,† and proceed regularly through the orders, so that when we have added any one order, we may know whether there are any from the lower order to add in with the higher one which we are to add next.

When we have gone through all the orders in this way we have the sum of the several numbers, since we have one number which is made up of all the others put together.

* Such solutions are frequently given in this treatise, not because the student is not supposed familiar with the reasons for the processes, but in order to show more clearly the identity of several processes usually treated as distinct.

† We shall frequently use the word "order" as the general term for both order and denomination.

In Ex. 1, having added the units column we have 29 units, which make 2 tens and 9 units. Hence we write the units under units column, and proceeding to tens column add the two tens in with this column.

In Ex. 2, having added the pints column we have 5 pints, which make 2 quarts and 1 pint. Hence we write the pints under pints column, and proceeding to quarts column add the 2 quarts in with this column.

In an exactly similar manner we proceed through the columns.

3. Add 75.641, 302.16, 428, 3.765, .042, 8.5, 70.31, and 6542.27.

Observe that the form of solution given above applies perfectly to this case.

4. Add 2472_8, 5640_8, 372_8, 12_8, 713_8, 4_8, 3111_8, and 6023_8 in the octary scale. *Sum*, $23011_8 = 9737_{10}$.

In this case we write like orders in the same column, beginning at the lowest order; having added any column, find how many of the next higher order the sum makes and how many remain; in short, the reasoning is in every particular the same as above.

75. It will be interesting for the student to observe that *with an addition table adapted to the octary scale* the process of "carrying" would be exactly the same as in the decimal scale, *i. e.*, we would write the right-hand figure of the sum of any column under that column, and add the left-hand figure (or figures) of this sum to the next column, with the same unconsciousness of dividing by the radix that we have in the common method. Thus, *with an octary addition table* we would say, in adding the first, or units column in the last example, 3, 4, 10, 13, 15, 17, 21, and writing the 1 under units add the 2 to the 2d order.

5. Add $4\frac{1}{5}$, $10\frac{2}{5}$, $6\frac{4}{5}$, $8\frac{3}{5}$, $1\frac{1}{5}$, and $\frac{3}{5}$.

$$\begin{array}{r} 4{*}1 \\ 10{*}2 \\ 6{*}4 \\ 8{*}3 \\ 1{*}1 \\ {*}3 \\ \hline 31{*}4 \text{ or } 31\tfrac{4}{5} \end{array}$$

Using the star to indicate that the figures at the right are 5ths, as we use the decimal point to indicate that they are 10ths, etc., we observe that we write 5ths in one column, units in another, etc., and adding the 5ths

find how many the sum makes of the next higher order, viz., units, etc., proceeding exactly as in the former cases.

6. Add $\frac{2}{3}$, $\frac{7}{12}$, $\frac{5}{8}$, $\frac{1}{2}$, $\frac{3}{4}$, and $\frac{5}{6}$.

We have here 2, 7, 5, 1, 3, and 5 to add, but they are all of different orders; hence, unless they can be reduced to the same order, they cannot be combined. They *may*, however, all be reduced to 24ths, and we then have 16, 14, 15, 12, 18, and 20 *twenty-fourths* to add. This gives 95 twenty-fourths, or $3\frac{23}{24}$, *i. e.*, 3 of the higher order, units, and 23 over. Thus we see that all the principles are the same as in simple addition.

$$\begin{aligned}
\tfrac{2}{3} &= 16 \text{ 24ths.}\\
\tfrac{7}{12} &= 14 \text{ 24ths.}\\
\tfrac{5}{8} &= 15 \text{ 24ths.}\\
\tfrac{1}{2} &= 12 \text{ 24ths.}\\
\tfrac{3}{4} &= 18 \text{ 24ths.}\\
\tfrac{5}{6} &= \underline{20} \text{ 24ths.}\\
&\ 95 \text{ 24ths.}
\end{aligned}$$

$$\text{or } \tfrac{95}{24} = 3\tfrac{23}{24}.$$

7. Add £2, \$7, 5 *fr.*, 1*s.*, 3 *M.*,* and 5*d.*

This example is exactly analogous to the last, the symbols £, \$, *fr.*, *s.*, *M.*, and *d.*, serving the same purpose as the denominators of the fractions in the 6th example. Now, as these numbers, 2, 7, 5, 1, 3, and 5 represent units of different orders, they cannot be combined in one sum unless they can be reduced to units of the same order. But we may reduce all to cents (or to any other of the denominations mentioned), just as in the 6th we reduced all to 24ths.

£2	=	973.3	cents.
\$7	=	700	"
5 *fr.*	=	96.5	"
1*s.*	=	24.3325	"
3 *M.*	=	71.4	"
5*d.*	=	$10.1385\frac{5}{12}$	"
		$1875.6710\frac{5}{12}$	cents.
	or	\$18.76—.	

8. Why cannot 7 *yd.*, \$20, 56 *lb.*, be added?

9. Add 36 *gr.*, 81 *pwt.*, 2 *lb.*, 30 *oz.*, explaining the process in accordance with the general principles of addition.

* Marks.

10. Add 3.5 meters, $6\frac{1}{2}$ feet, 482 centimeters, 286 inches, explaining as above.

Add the numbers in the last by reducing all to meters. Then by reducing all to feet. Then by reducing all to inches.

11. Add 5 *bbl.*, 40 *qts.*, liquid measure, and 6 *bush.*, 100 *pt.*, dry measure.

12. Add $\frac{7}{8}$ *cd.*, 156 *cu. ft.*, 185643 *cu. in.*, and $5\frac{3}{5}$ stere.

13. Add $300, £56, 2000 *fr.*, and 500 *M.*

14. Add 1.25 *mi.*, 7000 *rd.*, 4385 *ft.*

15. Add $\frac{3}{4}$ *lb.*, $5\frac{1}{2}$ *scruples*, $13\frac{3}{4}$ *gr.*

16. Add $\frac{5}{8}$ *sq. yd.*, $12\frac{2}{3}$ *sq. ft.*, 342.55 *sq. in.*

17. Add $\frac{4}{5}$, $\frac{2}{7}$, $\frac{3}{8}$, $\frac{5}{16}$, $\frac{10}{21}$, and $\frac{2}{3}$.*

18. Add 7.25 *ft.*, 3.82 *yd.*, 126.5 *in.*

19. Add 3.65, $\frac{3}{50}$, $1\frac{3}{4}$, 10.04, and $7\frac{3}{8}$.

20. Add $5\frac{1}{2}$ *mi.*, 488.3 *rd.*, $78\frac{1}{2}$ *ft.*, putting the sum in miles and decimals.

Literal Notation.†

75. 1. 5 times a certain number and 6 times the same number make how many times that number?

2. If a represents a certain number, how many times a are 5 times a, 6 times a, and 8 times a? Or, what is $5a + 6a + 8a$?

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* Such an example of course presumes a practical knowledge of the method of obtaining a common multiple; but this the student is supposed to have.

† Only the simplest elements of this subject can, or need be given here. We give just what is necessary to subsequent work.

3. Add $7x$, $3x$, $5x$, and $9x$. *Sum*, $24x$.

4. Show that $5ab + 6ab + 10ab = 21ab$.

5. Add $3a + 4b + 2c$, and $5a + 3b + 6c$.

$$\begin{array}{r} 3a + 4b + 2c \\ 5a + 3b + 6c \\ \hline 8a + 7b + 8c \end{array}$$

SOLUTION.—As $2c$ and $6c$ are similar terms (*i. e.*, alike as far as the letters are concerned), we can unite them. Hence we write one of them under the other. So also of $4b$ and $3b$, and of $3a$ and $5a$. This is the same as writing like orders or denominations in the same column. As there is no known relation between the number represented by c, and those by a and b, we have nothing here analogous to what is sometimes called "carrying" in the Arabic Arithmetic. We do not know whether $8c$ make any b's or not, for we do not know how many either c or b represents.

6. Add $5x + 4m + 3ab$, $10x + 6m + 5ab$, and $x + m + ab$. *Sum*, $16x + 11m + 9ab$.

x means the same as $1x$; m is *one* m, or $1m$, etc.

7. Add $5ax + 6by + m$, $12ax + 4by + 7m$, and $ax + 16by + 15m$. *Sum*, $18ax + 26by + 23m$.

8. Add $cd + 8ax + 10cy + 8ay$, $4cd + ax + cy + 15ay$, $3cd + 2ax + 5cy + 9ay$, and $7cd + 2ax + 9cy + 4ay$.

9. Add $3x^2$, $4x^2$, $7x^2$, and x^2.

x^2 means x multiplied by x, and is read "x square." But it is to be observed that for our present purpose it does not matter what x^2 means, so long as it means the same number in each case. Thus $4x^2$ and $8x^2$ is $12x^2$, no matter how many x^2 represents.

10. Add $2x^2 + 3x + 1$, $5x^2 + 4x + 5$, $x^2 + x + 4$, and $3x^2 + x + 2$. *Sum*, $11x^2 + 9x + 12$.

Can we unite into one term (expression) $3x$ and $2x^2$? Would it make $5x$, or $5x^2$?

11. Add $4x^3 + 6x^2 + 2x + 1$, $3x^3 + 8x^2$, $4x^3 + 6x$, $5x^2 + 10x$, $4x + 6$, and $x^2 + 6x$.

As above, it matters not what x^3 means, provided it means the same number each time. It is read "x cube," and means x multiplied by x and this product by x, *i. e.*, xxx.

$$\begin{array}{rrrr}
4x^3 + & 6x^2 + & 2x + & 1 \\
3x^3 + & 8x^2 & & \\
4x^3 & & + \ 6x & \\
 & 5x^2 & + \ 10x & \\
 & & 4x + & 6 \\
 & x^2 + & 6x & \\
\hline
11x^3 + & 20x^2 + & 28x + & 7
\end{array}$$

12. Add $5a^2 + 3a^3 + x$, $a^2 + 10a^3 + x$, $4a^2 + 7a^3 + 5x$, $31a^2 + 70a^3 + 19x$, $a^2 + x$, $a^3 + x$, $a^2 + a^3$, and $a^2 + a^3 + x$.

SECTION II.

MULTIPLICATION.

76. As in the last section we showed the essential unity of all those processes which we call Addition of numbers, in this we shall do the same concerning Multiplication.

77. ***A Product*** is a number which tells how many a certain number of times a given number makes.*

78. ***Multiplication*** is the process of finding the product of two numbers from a knowledge of the product of the digits, two and two, or by means of the Multiplication Table.*

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* These definitions, like those in (*72, 73*), are specially adapted to the Arabic Arithmetic. For definitions specially adapted to the Literal Arithmetic, see COMPLETE SCHOOL ALGEBRA (*81*).

It is to be observed that there are *three* ways of finding how many a certain number of times a given number makes, viz., by counting, by adding, and by the process we call Multiplication.

79. The idea and the process of Multiplication grow immediately out of Addition; thus 4 times 23 means $23+23+23+23$; $3\frac{1}{3}$ times 76 means $76+76+76+25\frac{1}{3}$; and $\frac{5}{8}$ times 35 means $4\frac{3}{8}+4\frac{3}{8}+4\frac{3}{8}+4\frac{3}{8}+4\frac{3}{8}$.

80. The ***General Problem**** in simple Multiplication is, *To find the product of two numbers, each represented by several digits,* as to multiply 578064 by 30286.

81. The ***Order of Development*** of the subject of Multiplication, *i. e.*, the succession of steps by which the solution of the General Problem is reached is,

1. *From our knowledge of addition we ascertain what the products of the digits taken two and two are, i. e., make the Multiplication Table.*

2. *Commit these products to memory, i. e., learn the Multiplication Table.*

3. *By means of these products learn to find the product of any two numbers, each represented by several digits.*

82. The ***Fundamental Principles*** on which the General Problem which we call Multiplication is based are the *six* following

* A Problem is something proposed to be done; hence "The General Problem" means the most comprehensive thing proposed to be done—that which includes all the others—the ultimate—the final purpose.

PROPOSITIONS.

1. *One number may be multiplied by another by multiplying the multiplicand by the parts of the multiplier and adding the products.*

2. *One number may be multiplied by another by multiplying the parts of the multiplicand by the multiplier, and adding the products.*

3. *One number may be multiplied by another by multiplying successively by all the factors of the multiplier; that is, by multiplying the multiplicand by one of the factors, and this product by another, and so on.*

4. *To multiply by* 10, 100, 1000, or 1 *with any number of* 0's *annexed, annex as many* 0's *to the multiplicand as there are in the multiplier; or in decimals remove the point a corresponding number of places to the right.*

5. *A multiplier is primarily an abstract number, and the product is of the same* ORDER *as the multiplier.*

6. *A multiplicand may be either abstract or concrete, and the product is of the same* KIND* *as the multiplicand.*

For a full and simple illustration of the first *four* of these propositions, see the author's ELEMENTS OF ARITHMETIC.

As to the 5th Proposition, it is manifestly absurd to attempt to use a concrete number as a multiplier; thus what could be meant by multiplying by 5 *pounds*, 4 *men*, or 7 *dollars*? A multiplier simply indicates a *number* of times which another number (abstract or concrete) is to be taken, and hence is mere number, *i. e.*, abstract. That the product is of the same order† as the multiplier becomes

---

* A careful reading of the sequel, it is thought, will justify this discrimination between *kind* and *order*. The advantage of the discrimination appears clearly in the case of denominate numbers and fractions.

† Observe that this conception can be affirmed of pure number; thus that 5 is 10's, 100th, or 7ths does not conflict with the notion that it is pure number.

evident when we consider that, if we multiply by units the result is units, if by tens the result is tens, if by thirds the result is thirds, *as far as the multiplier is concerned.* Thus to multiply any number by 1 ten (10) is to make it so many tens; whence to multiply it by 3 tens makes it 3 times as many tens. So to multiply 5 by 1 third is to take $\frac{1}{3}$ of each of the 5, making 5-thirds. Then to multiply 5 by 2 thirds is to make it 2 times 5, or 10-thirds, etc. (Thirds are orders, the same as tens or tenths.)

With reference to the 6th Proposition, it is evident that taking several times as much of a given quantity, or taking any part of it, does not change its nature; whence the product is of the same *kind* as the thing multiplied. Thus 4 times 5 *dollars* are 20 *dollars*, and $\frac{1}{2}$ of 5 *dollars* is $2\frac{1}{2}$ *dollars*.

Ex. 1. Multiply 8 by 3 hundreds.

To multiply 8 by 1 hundred is to make it 8 hundreds, whence to multiply by 3 hundreds is to make it 3 times as much, or 24 *hundreds* (Prop. 5).

2. Multiply 8 tens by 3 hundreds.

As above, the result is 24 hundreds of the kind multiplied, *i. e.*, hundreds of tens (Props. 5 and 6).

3. Multiply 6 by $\frac{3}{5}$.

To multiply 6 by 1-fifth is to make it 6-fifths (see above); whence to multiply it by 3-fifths is to make it 3 times as much, *i. e.*, 18-fifths.

4. Multiply 7 by .3, explaining as above.

5. Multiply 70 by .3, explaining as above.

6. Apply Props. 5 and 6 above to the multiplication of 483 by 6 hundreds.

6 *hundreds* times 3 *units* are 18 *hundreds** of *units*,† *i e.*, 18 hundreds. 6 hundreds times 8 tens are 48 hundreds* of tens,† *i. e.*, 48 thousands, etc.

* The *order*. † The *kind*.

7. Apply PROPS. 5 and 6 in multiplying 4 *yd.* 2 *ft.* 8 *in.* by 5. By 5 tens. By 5 hundreds.

When you multiply 2 *ft.* by 5 *tens* what is the *order* of the product? What the kind? What then is the product?

8. Apply the above principles in multiplying £4 8*s.* 11*d.* by 3. By .4. By .03.

£4	8*s.*	11*d.*
.4		
£1	15*s.*	6.8*d.*

£4 multiplied by 4 *tenths* gives 16 *tenths pounds;* the *order* being tenths, and the *kind* pounds. But 16 tenths pounds is £1.6, or £1 12*s.* Writing the £1 in its place, we carry forward the 12*s.* to be added to the next product. In like manner, .4 times 8*s.* makes 32 tenths shillings, which is 3.2*s.* = 3*s.* 2.4*d.*: to this adding the 12*s.* we have 15*s.*, which we write in shillings place, and have 2.4*d.* to carry to the pence place. .4 times 11*d.* = 4.4*d.*, to which adding the 2.4*d.* we have 6.8*d.*

Observe that in multiplying denominate numbers by a proper fraction it is convenient to begin with the highest instead of the lowest denomination. The reason is similar to that which makes it most convenient when multiplying by a whole number to begin with the lowest denomination. Let the student state it specifically, and illustrate it in the above example.

9. Multiply \$5 by $\frac{4}{5}$, showing the application of PROPS. 5 and 6.

See explanations of Examples 3 and 8 above.

10. Multiply $\frac{3}{5}$ by $\frac{2}{3}$, showing the application of PROPS. 5 and 6.

The *kind* of the multiplicand may be considered to be *fifths*, and the *order* of the multiplier *thirds.* Now 2 times 3 *fifths* are 6 *fifths*, and since the *order* of this result is *thirds*, we have 6 *thirds* of *fifths;* and as a third of a fifth is a fifteenth,* $\frac{3}{5} \times \frac{2}{3} = \frac{6}{15}$.

---

* In elementary teaching this fact would be illustrated by showing that if any thing is divided into 5 equal parts and then each of these into 3, the result is 15ths; and it may not be amiss to recur to it here.

11. Multiply 578 by 694, showing the application of PROPS. 1 to 4 above.

SOLUTION.—By PROP. 1 we obtain 694 times 578 by adding together 4 times 578, 90 times 578, and 600 times 578. By PROP. 2 we obtain 4 times 578 by taking 4 times 8, 4 times 7 tens, and 4 times 5 hundreds, adding the products as we go. By PROP. 3 we obtain 90 times 578 by first taking 9 times 578 and then 10 times this product, obtaining the latter by PROP. 4. In like manner we obtain 600 times 578. Finally, in accordance with PROP. 2, we add these partial products, and have 600 + 90 + 4, or 694 times 578.

```
   578
   694
  ----
  2312
 5202
3468
------
401132
```

12. Multiply 5.78 by 694, showing the application of the four principles as above.

13. Multiply £5 7*s.* 8*d.* by 694, explaining as above.

	£5	7*s.*	8*d.*
			694
4 times £5 7*s.* 8*d.*	21	10	8
90 times (10 times 9 times) £5 7*s.* 8*d.*	484	10	0
600 " (100 times 6 times) "	3230	0	0
694, or 600 + 90 + 4 times £5 7*s.* 8*d.*	£3736	0*s.*	8*d.*

14. Multiply 3.42 by 2.3, explaining as above.

The 3.42 may be considered as *hundredths*, and the 23 as *tenths*. Now 23 times 342 hundredths are 7866 *hundredths*, the *kind* being the same as the multiplicand. But as the *order* of the multiplier is 10ths, we have as a result 10ths of *hundredths*, or 1000ths. Hence $3.42 \times 2.3 = 7866$ thousandths, or 7.866.

15. Multiply $3\frac{5}{8}$ by 7.12.

16. Multiply $6\frac{4}{7}$ by $8\frac{3}{5}$.

17. Multiply 6 *bbl.* 18 *gal.* 3 *qt.* by $5\frac{3}{4}$.

18. Multiply 8 *lb.* 15 *pwt.* by .375.
19. Multiply 5 *cd.* 106 *cu. ft.* by $3\frac{1}{8}$.
20. Multiply 10° 17′ 20″ by 3.64.
21. Multiply 5 *A.* 81 *sq. rd.* 160 *sq. ft.* by 3.4.
22. Multiply 6.48 by .003. By 5.004.
23. Multiply .1 by $.0\frac{1}{8}$. By $\frac{2.5}{32}$.
24. Multiply $\frac{3.1}{4.3}$ by $7\frac{1}{8}$. By $\frac{.4}{56}$.
25. Multiply 5 *mi.* 4000 *ft.* by .3. By .015.

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26. Multiply 2300 by 5000, explaining the common method of neglecting the 0's in the process and annexing them to the product of the significant figures by PROPS. 5 and 6.

Call 23 the multiplicand, the *kind* being *hundreds*. Call 5 the multiplier, the *order* being *thousands*. The product, 115, is therefore thousands of hundreds.

27. Multiply 138000 by 750000, explaining as above.

The Literal Notation.

83. The Factors of a number are the numbers which multiplied together produce it.

Thus 3 and 5 are the factors of 15; 2, 5, and 7 are the factors of 70, etc.

84. A Power is a *product* arising from multiplying a number by itself.

Thus 3×3 makes 9, whence 9 is a power of 3. So 27 is a power of 3. 16, 64, and 256 are powers of 4.

85. The Degree of the power is indicated by the number of factors taken.

Thus 4, 8, 16, 32 are respectively the 2d, 3d, 4th, and 5th powers of 2.

86. As we have seen (**35**) when two or more letters representing numbers are written side by side, as ab, the product is indicated; but instead of writing a multiplied by a, thus, aa, it is written a^2. So aaa is written a^3, and $aaaa$ is written a^4, etc. The small figure written thus at the right and a little above is *one form* of an ***Exponent.**** a^2 is read "a square;" a^3 is read "a cube," or "a third power;" a^4 is read "a fourth power."

Ex. 1. What is the value of b^2 if $b = 4$? If $b = 8$? If $b = 10$? If $b = \frac{1}{2}$? If $b = .03$?

2. If $x = 6$, what is the value of x^2? Of x^3?

3. If $x = 4$, $y = 3$, and $a = 7$, show that $2a^2 + 3x^2y - 4ay = 158$.

4. What is the value of $a^2 + 2ab + b^2$, if $a = 4$, and $b = 20$? If $a = 30$, and $b = 6$?

5. What is the value of $x^2 - 2xy + y^2$, if $x = 60$, and $y = 9$? If $x = 10$, and $y = 4$?

6. What is the value of $a^3 + 3a^2b + 3ab^2 + b^3$, if $a = 20$, and $b = 5$? If $a = 40$, and $b = 7$?

---

* ***An Exponent*** is a number written a little to the right and above another number, and indicates

1st. *If a Positive Integer*, a *Power* of the number;

2d. *If a Positive Fraction*, the numerator indicates a *Power*, and the denominator a *Root* of the number;

3d. *If a Negative Integer or Fraction*, it indicates the *Reciprocal* of what it would signify if positive.

It is quite important that the pupil be guarded against the false impression that an *exponent* necessarily indicates a power. This impression, once made, is rarely entirely effaced.

7. What is the value of $a^3 - 3a^2b + 3ab^2 - b^3$, if $a = 600$, and $b = 40$? If $a = 6$, and $b = 2$? If $a = 8$ and $b = 5$?

---

87. Terms which have the same letters, affected with the same exponents, are called ***Similar.***

Ex. Point out the similar terms in the following: $5ax + 3by - 2ax + 10by^2 + 8b^2y - 10by - 6a^2x + 4ax - 30by + 756a^2x^2 + a^2x^2 + 4by^2 + 85b^2y + 5ax^2 - 7a^2x$.

Why are $-6a^2x$ and $+5ax^2$ not similar? Why are $85b^2y$ and $8b^2y$ similar?

---

8. Multiply $3ax + 2a^2y$ by ay.

$$\begin{array}{r} 3ax + 2a^2y \\ ay \\ \hline 3a^2xy + 2a^3y^2 \end{array}$$

SOLUTION.—This requires but the application of the ordinary principles (*82*). Thus, by PROP. 2, if we multiply the parts of the multiplicand, viz., $2a^2y$ and $3ax$, separately and add the products, we have the entire product. Now $2a^2yay$, or $2a^2ayy$,* signifies that the quantities are all multiplied together; but a^2a is a taken three times as a factor, and hence is written a^3 (*86*). So also yy is written y^2. Hence $2a^2y$ multiplied by ay is written $2a^3y^2$. In like manner $3ax$ multiplied by ay, or $3aaxy$, is written $3a^2xy$. Hence, adding the products, we have $3a^2xy + 2a^3y^2$.

9. Multiply $5ax + 3b$ by $2a^2$. *Prod.*, $10a^3x + 6a^2b$.

10. Multiply $3x^2 + xy + 2y^2$ by $5x^2$.
Prod., $15x^4 + 5x^3y + 10x^2y^2$.

---

* It is *assumed* that the order of the factors is immaterial, that is, that $8 \times 5 \times 7 = 5 \times 8 \times 7$, or $7 \times 8 \times 5$, etc. For a rigid demonstration of this principle, see the author's COMPLETE SCHOOL ALGEBRA (*85*), p. 51.

11. Multiply $3x^2y + 2x + 3$ by $7xy$.

Prod., $21x^3y^2 + 14x^2y + 21xy$.

12. Multiply $12a^3 + 2ab^2$ by $8a^2b$.

Prod., $96a^5b + 16a^3b^3$.

---

13. Multiply $3a^2x + 2ay$ by $2a + 3x^2$.

$$\begin{array}{r} 3a^2x + 2ay \\ 2a + 3x^2 \\ \hline 9a^2x^3 + 6ax^2y + 6a^3x + 4a^2y \end{array}$$

By PROP. 1 (*82*), if we multiply by the parts of the multiplier and then add the partial products, we shall obtain the entire product. Multiplying by $3x^2$ as above, we have $9a^2x^3 + 6ax^2y$. Then multiplying by $2a$, we have $6a^3x + 4a^2y$. As none of these terms are similar, we cannot unite them. All that can be done is to *indicate* the addition.

14. Multiply $a^2 + 2ab + b^2$ by $a + b$.

15. Multiply $2ax + 3by$ by $2ax + 3by$.

OPERATIONS.

$$\begin{array}{r} a^2 + 2ab + b^2 \\ a + b \\ \hline a^2b + 2ab^2 + b^3 \\ a^3 + 2a^2b + ab^2 \quad\quad \\ \hline a^3 + 3a^2b + 3ab^2 + b^3 \end{array} \qquad \begin{array}{r} 2ax + 3by \\ 2ax + 3by \\ \hline 6abxy + 9b^2y^2 \\ 4a^2x^2 + 6abxy \quad\quad \\ \hline 4a^2x^2 + 12abxy + 9b^2y^2 \end{array}$$

16. Multiply $a^2x + 2by$ by $2a^2x + by$.

17. Multiply $x + y$ by $x + y$.

18. Show that

$(a + b) \times (a + b) \times (a + b) = a^3 + 3a^2b + 3ab^2 + b^3$.

19. Show that $(x + 1) \times (x + 1) = x^2 + 2x + 1$.

20. Multiply $3a^2 + 2by + c$ by $3a^2y$.

21. Multiply $x^2 + x + 1$ by $x^3 + 1$.

SECTION III.

INVOLUTION.

88. *Involution* is the process of raising numbers to required powers. The number to be involved is called the *First Power*, or *The Root.*

As the power of a number is the product which arises from multiplying a number by itself any required number of times (*84*), Involution will be seen to be but a special case of Multiplication, viz., that in which the factors are equal.

Ex. 1. Involve 6 to the 3d power. To the 5th power.

2. Involve each of the digits to the 2d power. To the 3d power.

3. Commit to memory the 2d and 3d powers of the digits, *i. e.*, the squares and cubes.

Digits,	1,	2,	3,	4,	5,	6,	7,	8,	9.
Squares,	1,	4,	9,	16,	25,	36,	49,	64,	81.
Cubes,	1,	8,	27,	64,	125,	216,	343,	512,	729.

4. Square the following: 43, 602, 5.2, $8\frac{3}{4}$, $5\frac{1}{3}$, .034, 31.72, $\frac{5}{6}$, $\frac{7}{11}$, $2\frac{1}{2}$, 10, 100, 1000, 10000.

5. Cube the following: 84, 13.2, $1\frac{1}{2}$, .5, $\frac{2}{3}$, .08, 302, 10, 100, 1000.

---

6. Square 2×7; $3\times11\times7$; $2\times5\times8\times4$.

The square of 2×5 is 2×5 multiplied by 2×5, or $2\times5\times2\times5$, or $2^2\times5^2$, or 4×25, or 100.

The square of 2×7 is 2×7 multiplied by 2×7, *i. e.*, $2\times7\times2\times7$, or $2\times2\times7\times7$, or $2^2\times7^2$, or 4×49, or 196.

In like manner perform the others.

7. Cube, as above, 2×7; $3\times 11\times 7$; $2\times 5\times 8\times 4$.

8. Square the following: a, $2a$, ab, $5ab$, x^2, $4x^2$, $6x^2y$.

As above, the square of $5ab$ is $5ab\times 5ab$, or $5\times 5\times a\times a\times b\times b$, or $5^2\times a^2\times b^2$, or $25a^2b^2$. The square of x^2 is $x^2\times x^2$. But x^2 is xx; hence $x^2\times x^2$ is $xxxx$, or x^4.

9. Cube the following: c, $4c$, $5c^2$, y^2, x^3, $7x^2y$, $8ay^3$.

The cube of $7x^2y$ is $7x^2y\times 7x^2y\times 7x^2y$, or $7\times 7\times 7\times x^2\times x^2\times x^2\times y\times y\times y$, or $7^3x^6y^3$, or $343x^6y^3$. So explain the others.

10. Square $a+b$; $2ax+3b$; $1+y$; $2+cx^2$.

The square of $4x+3ay$ is $16x^2+24axy+9a^2y^2$.

11. Cube $a+b$; $4a+2b^2$; x^2+y^2; $1+x$.

The cube of $2x+3y^2$ is $8x^3+36x^2y^2+54xy^4+27y^6$.

12. Raise 2 to the 12th power.

The 12th power of 2 is composed of 12 factors, each 2. Now the square of 2 has two factors each 2, the square of the square has 4 such factors, the square of the 4th power has 8 such factors, and the 8th power multiplied by the 4th has 12. Hence $2^2=4$, $4^2=16$ (the 4th power of 2), $16^2=256$ (the 8th power of 2), $256\times 16=4096$ (the 8th power of 2 $\times$ by the 4th power) is the 12th power of 2.

13. Show that the 3d power of a number multiplied by the 3d power makes the 6th power. That the 4th power multiplied by the 5th power makes the 9th power.

14. What power of a number is obtained when the 2d power is multiplied by the 3d? The 5th by the 3d? The 4th by the 2d? The 8th by the 5th? Give the reason in each case.

15. Raise 23 to the 15th power by the fewest multiplications possible.

89. *Let the student write a rule for involving a number to any power by the principles elucidated in the last four exercises. Also for involving a monomial, Exs.* 8, 9.

PROPOSITIONS.

90. Prop. 1.—*The square of any number contains twice as many figures as the number itself, or 1 less than twice as many.*

Dem.—Considering the squares of any two consecutive numbers in the series 1, 10, 100, 1000, 10000, etc. (as 100 and 1000), we observe that as the square of each is 1 with double its number of 0's annexed, the square of the second contains *two* more figures than the square of the first. Now, as these numbers are the *least* numbers which can be represented by their respective number of figures, the square of any intermediate number (which contains the same number of figures as the less) contains as many figures as the square of the less, or one more. But the square of any one of these numbers contains 1 less than twice as many figures as the number itself.

Ex. 1. What is the least number which can be represented by 3 figures? What the least which can be represented by 4 figures? How many figures in the square of each? By how many figures are intermediate numbers represented? Then how many figures are there in the square of these intermediate numbers?

2. How many figures in the square of 3156? *Answer without multiplying.* Why must there be at least 7? Why can there not be 9? What is the least number which can be represented by 4 figures? What the least which can be represented by 5 figures?

3. Some number squared makes 18769. How many figures in it?

4. Some number squared makes 29855296. How many figures in it?

91. Prop. 2.—*The cube of any number contains three times as many figures as the number itself, or 1, or 2, less.*

Let the student write out a demonstration analogous to the above.

Ex. 1. The following are the cubes of certain numbers. How many figures in each of those numbers? 12167; 884736; 405224; 91125; 2460375; 11089567; 1191016; 17173512: 6372783864. Give the reason in each case, as under the last proposition.

92. Prop. 3.—*The square of any number made up of tens and units is the square of the tens, + twice the product of the tens by the units, + the square of the units.*

Dem.—Let a represent the tens and b the units; whence $a+b$ is the number. Now $(a+b)^2 = a^2+2ab+b^2$, which agrees with the statement.

Ex. 1. Square 48 in accordance with (Prop. 3).

The square of 4 tens is 16 hundreds; twice the product of 8 units and 4 tens is 64 tens; the square of 8 units is 64. Hence we have $(48)^2 = (4 \text{ tens } + 8 \text{ units})^2 = 1600+640+64 = 2304$.

2. Square 57 by (Prop. 3). Also 85. Also 79.

The square of 7 tens is	4900
Twice the product of 9 units and 7 tens is	1260
The square of 9 units	81
Therefore $(79)^2 = (7 \text{ tens } + 9 \text{ units})^2 =$	6241

3. Considering 345 as 34 tens + 5 units, square it by (Prop. 3). Also considering 4682 as 468 tens + 2 units, square it in like manner.

93. PROP. 4.—*The cube of any number made up of tens and units is the cube of the tens, + 3 times the square of the tens multiplied by the units, + 3 times the tens multiplied by the square of the units, + the cube of the units.*

DEM.—Let a represent the tens and b the units; whence $a+b$ is the number. Now $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, which agrees with the statement.

Ex. 1. Cube 65 by the above principle.

$$
\begin{array}{rrl}
 & a^3 = (6 \text{ tens})^3 & = 216000 \\
 & 3a^2b = 3\,(6 \text{ tens})^2 \times 5 & = 54000 \\
 & 3ab^2 = 3\,(6 \text{ tens}) \times 5^2 & = 4500 \\
 & b^3 = 5^3 & = \underline{125} \\
\text{Therefore} & 65^3 = (6 \text{ tens} + 5 \text{ units})^3 & = 274625
\end{array}
$$

2. Cube the following by (PROP. 4): 86; 43; 57; 31; 11.

3. Cube 283 by (PROP. 4), calling it 28 tens + 3 units. Also 1234, calling it 123 tens + 4 units.

Let the student observe that (PROPS. 3 and 4) are equally true whatever two parts a number is conceived to be made up of. Thus $(346)^2 = (3 \text{ hundreds} + 46)^2 = (3 \text{ hundreds})^2 + 2$ times 3 hundreds $\times\ 46 + (46)^2$.

4. Square 547, considering it as 5 hundreds + 47. Also 2345, considering it as 23 hundreds + 45. Also as 2 thousands + 345.

5. Cube 547 and 2345, considering the parts as in Ex. 4.

94. PROP. 5.—*If we separate the square of any number into periods by placing a point over the units figure, and one over each alternate figure to the left, the square of the highest order in the square root of this number is the greatest square in the left-hand period thus formed.*

DEM.—That is, 583696 being the square of a certain number, 49, the greatest square in 58, is the square of the highest order in the root, *i. e.*, the number of which 583696 is the square.

To prove this, we observe, *First*. The square of the figure of the highest order in any number (as the 7 in 7764) has twice as many orders below it in the square as there are places below it in the root, and hence falls in the left-hand period as described in the proposition.

Second. There remains to show that this square is the *greatest* square in the left-hand period. Now with any given figure as the highest order in the root, the remaining figures of the root cannot make the root as large as it would be with the next larger figure as the initial figure, even though the remaining figures at the right were 0's. But such a number as this last, squared, would produce only the next higher square in the left-hand period. Hence our supposed root being less, cannot produce this next higher square.

Ex. 1. Illustrate the above demonstration with the square 61559716, and its root 7846.

In what orders does the square of 7 thousands fall? What is the largest square in 61? What is the next larger square? What is the *least* number which, when squared, can give 64 in the fourth period according to this pointing?

2. Illustrate the above demonstration by squaring 648.

95. PROP. 6.—*If we separate the cube of any number into periods by placing a point over the units figure, and one over each third figure to the left, the cube of the highest order in the cube root of this number is the greatest cube in the left-hand period thus formed.*

Let the student write out the demonstration. It is entirely analogous to the preceding.

Ex. 1. Illustrate the demonstration of the above proposition by 12326391, which is the cube of 231.

2. Illustrate the above by cubing 67.

96. SYNOPSIS.

COMBINATIONS.

- ADDITION.
  - What. Sum.
  - Order of Development.
  - Fundamental Principles.
    - *Simple.*
    - *Compound.*
    - *Fractions.*
    - *Literal.*
- MULTIPLICATION.
  - What. Product.
  - Relation to Addition.
  - General Problem.
  - Order of Development.
  - Fundamental Principles.
    - *Simple.*
    - *Compound.*
    - *Fractions.*
    - *Literal.*
- INVOLUTION.
  - What. Relation to Multiplication.
  - Factor. Power. Degree of.
  - Square. Cube. Exponent.
  - Propositions.
    - 1.
    - 2.
    - 3.
    - 4.
    - 5.
    - 6.

CHAPTER IV.

RESOLUTION OF NUMBERS.

SECTION I.

SUBTRACTION.

[It is presumed that the *spirit* of the treatment of these subjects has been caught from the preceding chapter. Hence, in this the treatment is less detailed. But it is expected that teacher and pupil will carry out the method by careful attention to the suggestions given in this chapter, and reference to the preceding where that may be helpful.]

97. *Subtraction* is, primarily, the process of taking one number from another by means of a knowledge of the sums of the digits taken two and two.*

[The terms Subtrahend, Minuend, and Remainder are too familiar to need defining here.]

98. *The Order of Development* of the subject of Subtraction is,

1. *By a knowledge of the sum of the digits two and two, we learn to recognize the remainder when any digit is taken*

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* This definition is specially adapted to the Arabic Arithmetic. For a more comprehensive definition, see the author's COMPLETE SCHOOL ALGEBRA, p. 40. That the process we call subtraction is based on a knowledge of the sum of the digits taken two and two, is clear, since we say, "8 from 11 leaves 3, *because* 8 and 3 makes 11."

from any number not less than itself, but less than itself + 10.

2. *By means of this knowledge we learn to find the remainder when any number is taken from another.*

[See ELEMENTS OF ARITHMETIC.]

99. *The Fundamental Principles* of Subtraction are similar to those of Addition (*74*), the 3d being somewhat modified. (Let the student write them.)

Ex. 1. What is the remainder when 6841 is taken from 9563?

3. From 76432
Take 38526.

5. From 50476
Take 465.

2. What is the remainder when £6 8*s.* 4*d.* 1 *far.* is taken from £9 5*s.* 6*d.* 3 *far.*?

4. From 764.32
Take 385.26.

6. From 504.76
Take 465.

Let the student form solutions showing that the principles by which he solves the examples in the first column are quite adequate to the solution of those in the second. See solution of Ex. 1, 2, in Addition.

7. From 247.3 *ft.* subtract 5½ meters.

8. From 7 *gal.* subtract 2 quarts; 10 quarts.

9. From 70 subtract 8.

Let the student show that no principle is required in the solution of the 7th, or 8th, that is not required in the solution of the 9th. To subtract 2 *qt.* from 7 *gal.*, we may take 1 *gal.* and reduce it to quarts, etc. Can the 7th be solved in a similar way? The ordinary way of solving the 7th is what? Is there any different principle involved in this solution from that involved in taking *one* of the next higher order or denomination, and reducing it to the lower, as we usually do in Compound Subtraction?

10. Subtract 85 from 110; also 8.5 from 11; also $8\frac{2}{3}$ from 11; and show that the same principle applies in each case.

11. Subtract 32 francs from \$9.50; also $\frac{3}{7}$ from $\frac{4}{5}$; and show that the same principle is involved in each case.

12. Subtract 8 from 73; also $\frac{8}{11}$ from $7\frac{3}{5}$; and show that the same principle is involved in each.

What is there in simple subtraction which corresponds to reduction to forms having a common denominator in subtraction of fractions?

13. The minuend being 374 and the remainder 148, what is the subtrahend?

14. The subtrahend being $4\frac{2}{3}$ and the remainder $11\frac{1}{5}$, what is the minuend?

15. The remainder being 5 *cd.* 115 *cu. ft.* 349 *cu. in.* and the minuend 8 *cd.* 78 *cu. ft.* 769 *cu. in.*, what is the subtrahend?

16. From 7000 francs take 2000 marks.

100. The Difference between two numbers is the number of units which lie between them.*

ILL.—The difference between 7 days and 10 days is the number of days between the end of a period of 7 days and the end of a period of 10 days, *i. e.*, 3 days. The difference between 25° Fah. and 46° Fah. above 0, is the number of degrees on the thermometer scale between 25° and 46° above 0, *i. e.*, 21°. The difference between 15° above 0 and 10° below, is, in like manner, the number of degrees between these points, viz., 25°.

101. When two numbers are reckoned in the same direction from a common zero, their difference is found by subtracting one from the other.

* For a more complete exposition of this important subject, see the author's COMPLETE SCHOOL ALGEBRA, p. 41.

102. When two numbers are reckoned in opposite directions from a common zero, their difference is found by adding the numbers.

Ex. 1. What is the difference in longitude between two places, one in long. 42° 15′ 20″ E., and the other 31° 27′ 18″ E.?

2. What is the difference in longitude between two places, one in long. 42° 15′ 20″ E., and the other in 31° 27′ 18″ W.?

3. What is the difference between 26° 18′ 3″ north latitude, and 35° south latitude?

4. One morning the thermometer stood at 23° below 0, and the next at 35° above. What was the difference in temperature between the two mornings?

Literal Notation.

Ex. 1. What is the difference between 3 times a certain number and 5 times the same number, *i. e.*, between $3a$ and $5a$?

2. What is the difference between $7ay$ and $4ay$, *i. e.*, between 7 times a certain product and 4 times the same product?

3. What is $17a^2x - 13a^2x$? $9by^2 - 4by^2$?

4. What is the difference between $8ax + 7by$ and $3ax + 2by$?

SOLUTION.—The minuend being made up of the two parts, $8ax$ and $7by$, we can take $2by$ from either part, as is most convenient; hence we take it from the similar part, $7by$. For a like reason we take $3ax$ from $8ax$.

$$\begin{array}{r} 8ax + 7by \\ \underline{3ax + 2by} \\ 5ax + 5by \end{array}$$

5. From $3c^2x + 4ay + 10b^3$ take $c^2x + 4ay + 3b^3$.

Rem., $2c^2x + 7b^3$.

6. From x^3 take y^2.

As we are not supposed to know how many x or y represents, we cannot tell what the difference between x^3 and y^2 is, otherwise than by saying it is $x^3 - y^2$; hence this is called the remainder, or difference.

7. From $3a$ subtract $4b$.

8. From $5ax + 2c$ take $4by$.

As above, we can only write $5ax+2c-4by$, since not knowing the exact values of the letters we cannot tell the difference between $5ax+2c$ and $4by$ in a more specific way.

9. From x^3 take x^2. From x^2 take x. From a take 1. Subtract x from 1.

All that can be done in these cases is to *indicate* the subtraction. Why?

Positive and Negative.

103. ***Positive and Negative*** are terms primarily applied to concrete quantities which are, by the conditions of a problem, opposed in character.

ILL.—A man's *property* may be called positive, and his *debts* negative. Distance *up* may be called positive, and distance *down* negative. Time *before* a given period may be called positive, and *after*, negative. Degrees *above* 0 on the thermometer scale are called positive, and *below*, negative.

104. ***Terms having the + sign*** are called ***Positive,*** and those having the — sign, ***Negative.*** If no sign is written before a term the sign + is understood.

This is an immediate consequence of the definition of positive and negative, since in the polynomial $14ax-8ax+3ax-ax$ it is evident that the $14ax$ and $-8ax$ are opposed in character, *i. e.*, the $8ax$ is to be subtracted from $14ax$, and hence destroys part of it. So $14ax$ and $3ax$ are alike in character, while $-8ax$ and $-ax$ are opposed to them, though like each other.

Ex. 1. What is the sum of $-3ax$ and $-2ax$?

These terms being both negative, are alike in character, and being similar (87) can be united. The sum is $-5ax$.

2. What is the sum of \$$3ax$ and \$$2ax$?

3. What is the sum of $4by-2ax$ and $5by-4ax$?

Sum, $9by-6ax$.

4. What is the sum of $5x^2+2y^3-4cy$ and $3x^2+5y^3-2cy$?

---

5. What is the sum of $14ax$ and $-8ax$; *i. e.*, if they are united, what do they make?

As these are opposed in character, when united the $-8ax$ destroys $8ax$ from the $14ax$, and the result (which we call the sum) is $6ax$. This may also be seen thus, $14ax-8ax=6ax$.

It may seem strange that we call this the *sum*. But could we call it the *difference* between $14ax$ and $-8ax$? It *is* the difference between +*$14ax$ and +*$8ax$, but not the difference between $+14ax$ and $-8ax$.

6. The thermometer stands at a certain point and rises $5ax$ degrees, then falls $2ax$ degrees, then rises $10ax$ degrees, and again falls $4ax$ degrees. What is the sum or aggregate of all these variations?

As rise and fall are so opposed in their nature that one destroys the other, we may call rise + and fall −; hence we have to find the sum or aggregate of $+5ax$, $-2ax$, $+10ax$, and $-4ax$. This is $+9ax$, *i. e.*, on the whole the thermometer *rose* $9ax$ degrees.

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* These signs are introduced for emphasis.

7. One day the thermometer stood at $7ax$ degrees below 0 $(-7ax)$, and the next at $4ax$ degrees above 0 $(+4ax)$; what was the *difference* in temperature? *Ans.*, $11ax$.

105. From what has been shown about the addition of literal number we learn that,

1. ***In adding similar terms,*** *if the terms are all positive, the sum is positive; if all negative, the sum is negative; if some are positive and some negative, the sum takes the sign of that kind (positive or negative) which is in excess.*

2. ***Dissimilar terms*** *are not united into one by addition, but the operation of adding is expressed by writing them in succession, with the positive terms preceded by the* $+$ *sign and the negative by the* $-$ *sign.*

3. ***To add Polynomials,*** *write the expressions so that similar terms shall fall in the same column. Combine each group of similar terms into one term, and write the result underneath with its own sign. The polynomial thus found is the sum sought.*

8. Add the following polynomials: $2a - 8x^2$, $x^2 - 3a$, $-4a - 2x^2$, $4x^2 - a$. *Ans.*, $-6a - 5x^2$.

9. Add $2 - x + 4y^3$, $3 + 3x - y^3$, $-30 - x - 2y^3$, and $1 - 2x + 3y^3 - 10z$. *Ans.*, $-24 - x + 4y^3 - 10z$.

10. Add $3x + 5y - 6az$, $-2x - 8y - 9az$, $20x + 2y - 3az$, and $x - y + az - 4$.

Ans., $22x - 2y - 17az - 4$.

11. Add $3 - 2y + z$, $4y - 2z + 5$, $2 - z - y$, and $2z - y - 10$. *Ans.*, 0.

12. Add $3ax - 6$, $4ax + 2$, $6 + 3ax$, $7 - 2ax$, $ax + 1$

13. Add $4x - a$, $3a - x$, $4x + a$, $7a - 3x$.

14. Add $2 + 2z + 3z$, $3z + 8$, $5 - 2z$, $4z$.

15. $4x - 3x + 4 + x - 2x - 5 + 1 + 3x - 5x =$ what?

16. $2x + y + 9 - x - y - 9 + 3x + 10xy =$ what?

17. $5a + 3by - 4c + 2a - 5by + 6c + a - 4by - 2c$ = what?

18. $3a + x + 10 - 5a + 2x - 15 - 4a - 10x + 21$ = what?

19. $5a^2 - 3b + 4a^2 - 7b + 7a^2 + 3b - 5a^2 - 9b =$ what?

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106. The subtraction of Polynomials is based upon the principle that

Adding a negative quantity *destroys an equal positive quantity; and adding a positive quantity destroys an equal negative quantity.*

The truth of this principle is readily seen in the preceding examples and their solutions. Thus if we add $-2ax$ to $5ax$, the $2ax$ being opposed in character to the $5ax$, destroys $2ax$ out of $5ax$, leaving $3ax$. Again, if we wish to add $2ax$ to $5ax$, we may write $7ax-2ax$ for $5ax$. Now adding $2ax$ to $7ax-2ax$, we have $7ax-2ax+2ax$, or $7ax$, the $+2ax$ destroying the $-2ax$.

Ex. 1. What quantity added to $4ax - 3by + x^2$ will exactly destroy it?

2. What quantity added to $7c^2 - 4ax^2$ will exactly destroy it?

3. What polynomial added to any given polynomial exactly destroys it?

Ans., The same polynomial with the signs of all its terms changed.

4. If we wish to destroy $5ax$ out of $8ax$, what may we add to effect it? What to destroy $-3ay$ out of $-10ay$? What to destroy $-4a^2y$ out of $12a^2y$?

5. What must we add to any quantity to destroy $7ax-3a^2y$ out of it?

6. If we destroy out of a minuend the subtrahend, what remains?

From the above we see that

107. *To subtract one literal quantity from another,* *change the signs of each term in the subtrahend from + to −, or from − to +, or conceive them to be changed, and add the result to the minuend.*

REASONS.

Why do you change the signs of the subtrahend? *Ans.*, To get a quantity which added to the minuend will destroy out of it an amount equal to the given subtrahend, according to (***106***).

Why do you add the subtrahend with its signs changed to the minuend? *Ans.*, Because, as the minuend is the sum of the subtrahend and remainder, if we destroy the subtrahend from out the minuend, we have left the remainder.

7. From $8ay-3bx^2$ subtract $3ay+2bx^2$.

SOLUTION.—If we destroy the value of the subtrahend out of the minuend, what is left is the remainder sought. Now to destroy $3ay+2bx^2$ out of any quantity, we have but to add $-3ay-2bx^2$ (***106***). Adding this, we have the remainder, $5ay-5bx^2$.

OPERATION.

To	$8ay-3bx^2$
Add	$-3ay-2bx^2$
	$5ay-5bx^2$

8. From $7ax+2b^2y-3c$ subtract $-2ax+5b^2y-c$.

9. From $8ab-6a^2z^2-a^3y$ take $11a^2z^2+3a^3y$.

10. From $3a-2b$ take $b-2a$.

11. From $4-3xy$ take $xy-1$.

108. *The Law of the Signs in Multiplication is that,* *If the multiplier and multiplicand have* LIKE *signs the product is* $+$ *; but if they have* UNLIKE *signs the product is* $-$.

DEM.—1. To multiply $+a$ by $+b$. By (*82*, 5) the multiplier, b, is at first to be conceived as abstract, *i. e.*, a mere number; and by (*82*, 6) the product is like the multiplicand. Hence b times $+a$ gives $+ab$. Now the sign of b being $+$ shows that this product is to be taken additively, *i. e.*, with its own sign. Hence $(+a)\times(+b) = +ab$.

2. To multiply $-a$ by $-b$, first considering the multiplier as abstract, we have b times $-a$ equal to $-ab$ (*82*, 6). Now the sign of b being $-$ shows that this product is to be taken subtractively, *i. e.*, *with its sign changed.* Hence $(-a)\times(-b) = +ab$.

3. To multiply $-a$ by $+b$. As before, b times $-a$ gives $-ab$ (*82*, 6); and the sign of b being $+$ shows that this product is to be taken additively, *i. e.*, with its own sign. Hence $(-a)\times(+b) = -ab$.

4. To multiply $+a$ by $-b$. b times $+a$ is $+ab$, and this taken subtractively becomes $-ab$. Hence $(+a)\times(-b) = -ab$.

Ex. 1. Multiply $5ac - 3a^2 + by$ by $4a^2y$. By $-4a^2y$.

$$\textit{Prods.,}\ \begin{cases} 20a^3cy - 12a^4y + 4a^2by^2, \text{ and} \\ -20a^3cy + 12a^4y - 4a^2by^2. \end{cases}$$

2. Multiply $3a - 2b$ by $3a + 2b$. *Prod.,* $9a^2 - 4b^2$.

3. Multiply $3a - 2b$ by $3a - 2b$.

Prod., $9a^2 - 12ab + 4b^2$.

4. Multiply $x^2 - 2xy + y^2$ by $x - y$.

Prod., $x^3 - 3x^2y + 3xy^2 - y^3$.

5. Raise $a - b$ to the 4th power.

Result, $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$.

6. Multiply $1 - r$ by $1 - r$. By $1 + r$.

7. Raise $1 + r$ to the 6th power. $1 - r$ to the 5th power.

8. Multiply $1 + rp$ by $\frac{r}{p}$. *Prod.,* $\frac{r}{p} + r^2$.

9. Multiply $ar - a$ by $1 - \frac{a}{r}$. *Prod.*, $ar - a - a^2 + \frac{a^2}{r}$.

10. Multiply $\frac{b^2}{a^2} - x^2y^2$ by $\frac{a^2}{x^2}$.

Prod., $\frac{b^2}{x^2} - a^2y^2$, or $\frac{b^2 - a^2x^2y^2}{x^2}$.

11. Multiply $\frac{a}{b}$ by $\frac{x}{y}$. *Prod.*, $\frac{ax}{by}$.

12. Multiply $1 - \frac{a}{b}$ by $1 + \frac{x}{y}$, both by reducing the mixed numbers to improper fractions, and without.

Prod., $\frac{by - ay + bx - ax}{by}$, or $1 - \frac{a}{b} + \frac{x}{y} - \frac{ax}{by}$.

13. Show that the two results in the last are identical.

The Parenthesis.

Ex. 1. Perform the operation indicated by $a(a - b)$. By $(a - b)(a + b)$. By $(a + c)^2$.

2. Perform the operation indicated by $a^2 + 2ax + x^2 - (a^2 - 2ax + x^2)$. *Result*, $4ax$.

The quantity in the parenthesis is to be taken together as affected by the sign before it (*55*); *i. e.*, in this case $a^2 - 2ax + x^2$ is to be subtracted from the preceding.

3. What is the value of $5ax - 2(c + b)$, if $a = 6$, $x = 2$, $c = 3$, and $b = 5$? *Ans.*, 44.

4. What is the value of $3x^2 - (a^2 - b^2) + 2c$, with the same values as in the last? *Ans.*, 7.

5. What is the value of $14 - (5 - 2) - 4(3 + 5) + 6(7 - 3)$? *Ans.*, 3.

6. What is the value of $(9 - 3)(7 + 2)$? Of $3(4 - 2) - 5(7 - 1) \times \frac{2}{3}$? *Ans. to last*, -14.

7. What is the value of $\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\times\frac{2}{3}\div\frac{1}{2}$? What of $\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)\times\frac{2}{3}\div\frac{1}{2}$?

Ans. $1\frac{2}{3}$, and 2.

8. What is the value of $5-\frac{4+3}{11}$? Is $5-\frac{4+3}{11}$ the same as $\frac{55-4-3}{11}$? Why?

9. What is the value of $3a-\frac{2a^2-5x}{4b}$, if $a=5$, $x=10$, and $b=3$?

$3a-\frac{2a^2-5x}{4b}=\frac{12ab-2a^2+5x}{4b}=\frac{180-50+50}{12}=15$. So, also, $\frac{2a^2-5x}{4b}=\frac{50-50}{12}=\frac{0}{12}=0$. Hence the given expression is the same as $3a$, *i. e.*, 15, the fraction being 0.

109. *When several terms are enclosed in a parenthesis, or other equivalent symbol,* PRECEDED BY A — SIGN, *if the parenthesis is dropped the signs of all the terms within must be changed.*

The reason for this is that the — sign shows that the polynomial within the parenthesis is to be subtracted. Hence by (***107***) we are to change the signs of its terms and add the result.

This principle is equally true if but *one* term is enclosed; thus $a-(b)$, *i. e.*, $a-(+b)=a-b$, and $a-(-b)=a+b$.

10. Remove the parentheses from $3ax-(c-15a)$. From $4y+(3-ab)$. From $(6ay-3cd)-5$. From $7ax+2by-(10d+4)$. After having removed the parentheses from each, evaluate the results for $a=10$, $x=6$, $c=8$, $d=1$, $y=3$, $b=2$.

Results, 187, — 5, 151, and 418.

SECTION III.

DIVISION.

110. Division is a process of finding how many times one number is contained in another.

The process called Division is the converse of Multiplication; *i. e.,* by means of a knowledge of the products of numbers, we find one factor when the other factor and the product are given.

Division also enables us to separate a number into any number of equal parts, and find how many there are in one of these parts.

The primary notion of division is, without doubt, that of separating a number into equal parts. The word itself makes this evident. This separation may be made for either of two purposes, viz., to ascertain how many such parts there are in the number, or to find how many there are in one of the parts.

111. The problem of Division may be solved by Subtraction; but the process which we call Division is not based, primarily, upon Subtraction, but upon Multiplication.

[The terms Dividend, Divisor, Quotient, and Remainder are too familiar to need defining here.]

Ex. 1. Divide 2176 by 68, and show the relation of the process to Multiplication and Subtraction.

$$\begin{array}{r|l} 68\,) & 2176\,(\,32 \\ & \underline{204} \\ & 136 \\ & \underline{136} \end{array}$$

In the first place we find how many times 68 is contained in 217 (tens) by finding how many times 6 is contained in 21. To ascertain this latter fact we appeal alone to our knowledge of products, *i. e.,* to

multiplication. Thus, inasmuch as we know that $3 \times 6 = 18$, and $4 \times 6 = 24$, we know that 6 is contained in 21 3 times. Hence *it is made probable* that 68 is contained in 217 (tens) 3 (tens) times. To test this point we *multiply* 68 by 3 (tens), and find that it *is* contained 3 (tens) times. All of this part of the work is seen to be based entirely upon Multiplication.

Again, in order to find how much of the 2176 remains undivided, we *subtract* 3 (tens) times 68 from it, finding that 68 is contained in 2176 3 (tens) times, with a remainder of 136. This part of the process is based upon the principle that *one number is contained in another as many times as it can be subtracted from it in succession.* This principle expresses the relation which Division sustains to Subtraction.

The work is carried forward on exactly the same principles. Thus we find (approximately) how many times 68 is contained in 136 by finding how many times 3 is contained in 13, and this is determined by our knowledge of products, etc.

112. The Order of Development of the subject of Division is,

1. *To observe from our knowledge of the products of the digits two and two, what the other digit is when the product and one of the digits is given, i. e., to see the Division Table in the Multiplication Table.*

2. *To make the first three (at least) of the following principles practically familiar.*

3. *To apply the knowledge gained in the two following steps to the solution of the GENERAL PROBLEM, i. e., to divide a number represented by several digits by another number represented by several.*

113. The Principles upon which the process called Division is founded are,

1. *Whatever number of times a given divisor is contained in a given dividend, this divisor is contained in twice this dividend, twice as many times; in 3 times this dividend,*

3 times as many times; in 10 times this dividend, 10 times as many times, etc.

2. *We may find how many times a given divisor is contained in a given dividend, by finding how many times it is contained in all the parts of the dividend and adding the results together.*

3. *If a given divisor is contained in any dividend a certain number of times with a certain remainder, it is contained in 2 times that dividend 2 times as many times with 2 times as great a remainder, in 3 times that dividend 3 times as many times with 3 times as great a remainder, in 10 times as great a dividend 10 times as many times with 10 times as great a remainder, etc.*

4. *If the dividend and divisor are considered as representing numbers of any particular* KIND* *(as concrete numbers), they must represent numbers of the* SAME KIND.

5. *If the dividend is considered as representing some particular* KIND, *and the divisor is abstract, the quotient is of the same kind as the dividend.*

For illustrations of the first three see ELEMENTS OF ARITHMETIC (*71, 72, 73*).

The 4th and 5th are self-evident, since while it is intelligible to ask how many times $5 is contained in $30, it is absurd to inquire how many times $5 is contained in 30 *feet*. So also if a number be divided into equal parts, the parts are of the same kind as the whole.

REMARK.—It will be remarked that the conception of division as finding how many times one number is contained in another, is the only one consistent with the idea of dividing one concrete number by another; while the conception of division as a method of finding one of the equal parts of a concrete number requires that the divisor be conceived as abstract.

* This word is used, as heretofore, to embrace somewhat more than the term *concrete*. Thus 24-*thirds* and 3-*sixths* are not considered as concrete, yet the numbers 24 and 3 represent different *kinds*, in the sense in which we use this word.

Ex. 1. Divide 7682 by 9.

2. Divide 76 *bbl.* 8 *gal.* 2 *qt.* by 9.

3. Divide $125 by 5.

4. Divide $\frac{125}{327}$ by 5.

5. Divide $7 by 5 francs.

6. Divide $\frac{7}{3}$ by $\frac{5}{4}$.

The object aimed at in these examples is to show the essential identity of all the processes we call Division, and also that the principles given in (***113***) apply to all cases. Thus in 1 and 2 we commence the division at the left hand in each case, and for the same reason, viz., that the remainders which arise from dividing each part may the more readily be combined with the undivided part. In each case we obtain the entire quotient by adding the partial quotients obtained by dividing the parts of the dividend, which addition is facilitated by beginning the division with the highest order.

The 3d and 4th are both covered by (PRIN. 5). Let the student show how.

Again, observe the application of (PRIN. 4) to 5 and 6. As the $7 and 5 francs have to be reduced to like denominations, as 700 *cents* and 96.5 *cents*, before the division can be effected, so 7 *thirds* and 5 *fourths* have to be reduced to like denominations, as 28 *twelfths* and 15 *twelfths*. Thus $7÷5 francs is 700÷96.5, and $\frac{7}{3} \div \frac{5}{4}$ is 28÷15.

7. Divide 3 *yd.* 5 *ft.* 2 *in.* by $\frac{2}{3}$ of a meter; and $4\frac{3}{8}$ by .25, showing the common application of (PRIN. 4).

8. Divide 7 *lb.* 12 *oz.* *Av.* by 1 *lb.* 20 *pwt.* Troy.

What principle applies?

9. Divide $\frac{3}{7}$ of a ton by $\frac{3}{5}$.

What principle applies?

10. Divide $\frac{3}{16}$ by $\frac{4}{7}$. 43 by $\frac{3}{5}$.

While the analysis suggested above is best calculated to exhibit the application of general principles, the following has the advantage of giving a direct explanation of the common process of dividing by a fraction. $\frac{4}{7}$ is contained in 1, $\frac{7}{4}$ times; hence it is contained in $\frac{3}{16}$, $\frac{3}{16}$ of $\frac{7}{4}$ times, or $\frac{21}{64}$. So also $\frac{3}{5}$ is contained in 1, $\frac{5}{3}$ times; hence it is contained in 43, 43 times $\frac{5}{3}$ times, or $\frac{215}{3} = 71\frac{2}{3}$.

11. Show that $\frac{3}{5}$ is contained in 1, $\frac{5}{3}$ times. That $\frac{4}{7}$ is contained in 1, $\frac{7}{4}$ times.

How many times is $\frac{1}{8}$ contained in 1? How does (PRIN. 4) apply to this?

12. Divide $\frac{3}{8}$ of a cord by $5\frac{1}{3}$ *cu. ft.*

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13. Divide 12.5 by .5. By .05. By 5.

Perhaps the most simple exposition of the pointing off of the quotient in division of decimals, is found in the application of (PRIN. 4); and the method is quite as convenient in practice as any. Thus, in order to divide 12.5 by .5, both must be reduced to the same kind. Now 12.5 is 125 *tenths*, and .5 is 5 *tenths*. Hence our quotient is $125 \div 5 = 25$. Again, 12.5 is 1250 *hundredths*, and .05 is 5 *hundredths*. Hence $12.5 \div .05 = 1250 \div 5 = 250$. So also $12.5 \div 5 = 125 \div 50 = 2.5$.

The practical rule deduced from this analysis is as follows:

114. ***To divide when dividend or divisor or both contain decimals,*** *make the number of decimal places equal in each, and dropping the decimal point from both (or disregarding it), divide as in whole numbers.*

14. Divide .78 by 3. By 30. By .3. By .003.

The rule gives $78 \div 300$. $78 \div 3000$. $78 \div 30$. $780 \div 3$.

15. Divide 17.63 by 2.4. By .12. By 52. By .251. By 1.0065.

The latter becomes $176300 \div 10065$. We begin to produce decimals in the quotient, when we begin to annex 0's to the remainder.

115. PROP.—*We may divide by a composite number, by dividing in succession by its factors.*

DEM.—Thus to divide by 15 is to get $\frac{1}{15}$ of a number. Hence, if we first get $\frac{1}{3}$ by dividing by 3, and then $\frac{1}{5}$ of this result by dividing by 5, we have $\frac{1}{5}$ of $\frac{1}{3}$, or $\frac{1}{15}$ of the number. This method of reasoning can be extended to any number of factors into which we may be able to resolve the divisor, and hence applies to all cases.

For a more elementary demonstration, or illustration, see ELEMENTS OF ARITHMETIC, p. 105.

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116. Let the student give the method of dividing by 10, 100, 1000, etc., and the reason for the method. Also for dividing when the divisor consists of other digits than 1, with 0's at the right.

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117. The following propositions are direct consequences of the definition of division:

PROP. 1.—*Dividend and divisor may both be multiplied or both be divided by the same number without affecting the quotient.*

This may be illustrated thus: If a given number of apples are divided among any number of boys, each boy will receive just the same number as if twice or thrice as many were divided among twice or thrice as many boys, or as if $\frac{1}{2}$ or $\frac{1}{3}$ as many were divided among $\frac{1}{2}$ or $\frac{1}{3}$ as many boys.

PROP. 2.—*If the dividend be multiplied or divided by any number, while the divisor remains the same, the quotient is multiplied or divided by the same.*

Student illustrate this and the two following in like manner as the above.

PROP. 3.—*If the divisor be multiplied by any number while the dividend remains the same, the quotient is divided by that number; but if the divisor be divided, the quotient is multiplied.*

PROP. 4.—*The sum of the quotients of two or more quantities divided by a common divisor, is the same as the quotient of the sum of the quantities divided by the same divisor.*

To illustrate this, consider that as 2 goes into 8, 4 times, and into 6, 3 times, it will go into 8 and 6, or $8+6$, 4 times + 3 times, or 7 times, etc. This is in substance the same principle as (2, ***113***).

PROP. 5.—*The difference of the quotients of two quantities divided by a common divisor, is the same as the quotient of the difference divided by the same divisor.*

This can be illustrated as the last.

118. ***Cancellation*** is the striking out of a factor common to both dividend and divisor, and does not affect the quotient, as appears from (PROP. 1) above.

119. PROBLEM 1.—*To show that the conception of a fraction as representing a certain number of equal parts, is consistent with the conception of it which considers the numerator as a dividend and the denominator its divisor.*

SOLUTION.—Since upon the first conception the denominator shows into how many equal parts a thing is conceived to be divided, it also shows how many such parts make a unit. But the numerator shows how many such parts are represented by the fraction. Hence, dividing the number of parts given by the number which it takes to make a unit, shows how many units there are in the expression, which is its value.

REMARK.—It will be observed that this is the familiar argument of *Reduction Ascending.*

120. PROB. 2.—*To demonstrate all the operations in common fractions on the above principles, i. e., regarding the fraction as an unexecuted problem in division, viz.:*

1. Reduction to lower or higher terms.
2. Reduction of Improper Fractions to whole or mixed numbers.
3. The converse of the last.
4. Reduction to forms having a C. D.
5. Addition and Subtraction.
6. Multiplication and Division.
7. Involution and Evolution.

We indicate the method in reference to 4 and 6; let the student give it in each of the other cases.

Thus, to reduce $\frac{5}{7}$ and $\frac{3}{4}$ to forms having a common denominator: $\frac{5}{7}=\frac{5\cdot 4}{7\cdot 4}=\frac{20}{28}$, and $\frac{3}{4}=\frac{3\cdot 7}{4\cdot 7}=\frac{21}{28}$, by (PROP. 1), since the numerator is the dividend, the denominator the divisor, and the value of the fraction is the quotient.

In explaining addition and subtraction, observe the application of (PROPS. 4 and 5).

To multiply $\frac{3}{5}$ by $\frac{4}{11}$. First multiplying by 4, according to PROP. 2 (or 3, if we can), we have $\frac{12}{5}$. But as our multiplier was not to be 4, but 4 divided by 11, and as dividing the multiplier divides the product, our product $\frac{12}{5}$ is 11 times as large as it should be. Hence we divide it by 11, by PROP. 2 (or 3, if we can),* obtaining $\frac{12}{55}$.

To divide $\frac{4}{7}$ by $\frac{5}{9}$. First dividing by 5, according to PROP. 3 (or 2, if we can), we have a quotient, $\frac{4}{35}$, which is only $\frac{1}{9}$ as large as it should be, since our divisor was 5 divided by 9, and divding the divisor divides the quotient (PROP. 3). Hence we multiply $\frac{4}{35}$ by 9, either by multiplying the 4 or dividing the 35, as is most convenient (PROP. 2, or 3).

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* If we had $\frac{33}{8}$ to multiply by $\frac{4}{11}$, it would be more elegant to multiply by 4 by dividing the denominator (PROP. 3), and divide by 11 by dividing the numerator by 11 (PROP. 2).

Literal Notation.

Ex. 1. Divide $12a^2x$ by $3a$.

Since rejecting common factors from divisor and dividend does not alter the quotient (*118*), we have $12a^2x \div 3a = 4ax \div 1$, or $4ax$; as 3 is a factor in both and may be stricken out, and also a.

2. Divide $12a^2x$ by $3ay$.

This may be written $\frac{12a^2x}{3ay}$, whence, striking out (canceling) the common factors 3 and a, we have $\frac{4ax}{y}$. Or, we may explain thus: $12a^2x \div 3ay = 4ax \div y = \frac{4ax}{y}$, the operation of dividing by y being simply *indicated*, inasmuch as there is no like factor in the dividend.

121. *Let the student write a rule for dividing monomials in such cases as the above.*

***122. The Law of the Signs in Division** is that* LIKE *signs in divisor and dividend give + in the quotient, and* UNLIKE *signs —.*

This is a direct consequence of the law of signs in multiplication. Let the student show how this law grows out of the law that if multiplier and multiplicand have like signs the product is +, and if unlike, —.

3. Divide $18a^3x^2y$ by $-12a^2xy$.

Quot., $-\frac{3ax}{2}$, or $-\frac{3}{2}ax$.

4. Divide $-20x^2y^3$ by $4xy^2$. *Quot.*, $-5xy$.

5. Divide $-6axy^2$ by $-15a^2y$. *Quot.*, $\frac{2xy}{5a}$.

6. Divide $-13ay$ by $17bx$. *Quot.*, $-\frac{13ay}{17bx}$.

7. Divide $12a^2b$ by $4ab^2$.
8. Divide $-16a^2x$ by $4ax^2$.
9. Divide $144a^3b^2x$ by $-24a^2b^4$.
10. Divide $a^3b^2c^3$ by $a^3b^2cx^2$.
11. Divide a by b.
12. Divide $-16a^2$ by $4a^2x$.
13. Divide $-6mx^3$ by $-24nx^4$.
14. Divide $256a^5b^4c^3$ by $16a^5b^2c^3mn$.
15. Divide $15ax^2y^2$ by $45a^2bx^3y^2$.
16. Divide $39496a^3x^3$ by $69118a^4x^2y$.
17. Divide $58760x$ by $66105x^3y$.
18. Divide $25194a^3x$ by $88179a^2x^3$.

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19. Divide $12a^2x + 18a^3x^2$ by $3ax$.

By (Prop. 4, ***117***), or, what is the same thing, by (2, ***113***), we can divide the parts of the dividend, $12a^2x$ and $18a^3x^2$, separately, and add the quotients.

OPERATION.

$$3ax\,)\,\underline{12a^2x + 18a^3x^2}$$
$$4a \quad + \quad 6a^2x$$

20. Divide $12a^2x - 18a^3x^2$ by $3ax$.

This operation may be explained by (Prop. 5, ***117***); or using "add" in its most general sense—uniting quantities with their own signs, and observing the law of signs in division, it is the same as the last.

OPERATION.

$$3ax\,)\,\underline{12a^2x - 18a^3x^2}$$
$$4a \quad - \quad 6a^2x$$

21. Divide $12a^2x - 18a^3x^2$ by $-3ax$.

Quot., $-4a + 6a^2x$.

22. Divide $16am^2x - 8a^2mx^2 + 4a^3m^3x^3$ by $4amx$.

Quot., $4m - 2ax + a^2m^2x^2$.

123. *Let the student write a rule for dividing a polynomial by a monomial, as in the above examples.*

23. Divide $10xz + 15xy$ by $5x$.
24. Divide $15ax - 27x$ by $3x$.
25. Divide $18x^2 - 9x$ by $9x$.
26. Divide $abc - bcd - bcx$ by $-bc$.
27. Divide $3x + 6x^3 + 3ax - 15x$ by $3x$.
28. Divide $3abc + 12abx - 9a^2b$ by $3ab$.
29. Divide $40a^3b^2 + 60a^2b^2 - 17ab$ by ab.
30. Divide $15a^2bc - 10acx^2 + 5ad^2c$ by $-5ac$.
31. Divide $20ax + 15ax^2 + 10ax - 5a$ by $5a$.
32. Divide $12xy - 9x^2y^2 - 3ax$ by $3xy$.

Quot., $4 - 3xy - \dfrac{a}{y}$.

33. Divide $5a^2x - 4a^3x^3$ by $7a^4x^3$. *Quot.*, $\dfrac{5}{7a^2x^2} - \dfrac{4}{7a}$.

---

34. Divide $\dfrac{x}{y}$ by $\dfrac{a}{b}$. *Quot.*, $\dfrac{bx}{ay}$.

The process is the same as in ordinary division of fractions; thus

$$\frac{x}{y} \div \frac{a}{b} = \frac{x}{y} \times \frac{b}{a} = \frac{bx}{ay}.$$

35. Divide $\dfrac{12a^2x^3}{35b^3y^3}$ by $\dfrac{6ax^3}{7b^2y^3}$. *Quot.*, $\dfrac{2a}{5b}$.

36. Divide $\dfrac{1-r}{1+r}$ by $(1-r)^2$.

$$\frac{1-r}{1+r} \div (1-r)^2 = \frac{1-r}{(1+r)(1-r)^2} = \frac{1}{(1+r)(1-r)} = \frac{1}{1-r^2}.$$

37. Reduce $5a + \frac{3b}{2x}$ to the form of an improper fraction.

Result, $\frac{10ax + 3b}{2x}$.

Process same as in common arithmetic.

38. Reduce $5a - \frac{3b}{2x}$ to the form of an improper fraction.

Result, $\frac{10ax - 3b}{2x}$.

39. Divide $1 - \frac{x}{y}$ by $1 + \frac{x}{y}$. *Quot.*, $\frac{y - x}{y + x}$.

40. Divide $\frac{2a}{3b} - \frac{x}{y}$ by $\frac{a}{b} + \frac{2x}{3y}$. *Quot.*, $\frac{2ay - 3bx}{3ay + 2bx}$.

SECTION IV.

EVOLUTION.

124. A Root is one of the equal factors into which a number is conceived to be resolved. The *Square Root* of a number is one of *two* equal factors into which the number is conceived to be resolved. The *Cube Root* is one of three equal factors.

125. The *Radical* or *Root Sign* is $\sqrt{}$. When written thus $\sqrt{25}$, it indicates that the square root of 25 is to be taken; that is, that 25 is to be resolved into 2 equal factors, and one of them taken. To indicate the cube root, 3 is written in the sign. Thus $\sqrt[3]{125}$ means the cube root of 125. It is 5.

In like manner the 4th root is one of the 4 equal factors which compose a number, and is indicated thus $\sqrt[4]{}$; the 5th root thus $\sqrt[5]{}$, etc.

126. *Evolution* is the process of extracting roots of numbers.

As evolution is the process of finding one of a certain number of equal factors which compose a number, it is but a process of factoring—resolving a number into equal factors.

Ex. 1. Show what $\sqrt{16} =$. $\sqrt[5]{32} =$. $\sqrt[3]{1728} =$.

2. What is the square root of 1764?

$$\begin{array}{r|l} 2 & 1764 \\ 2 & 882 \\ 3 & 441 \\ 3 & 147 \\ 7 & 49 \\ & 7 \end{array}$$

Resolving 1764 into its Prime Factors* we find them to be 2, 2, 3, 3, 7, 7. Hence,

$$2 \cdot 3 \cdot 7 \times 2 \cdot 3 \cdot 7, \text{ or } 42 \times 42 = 1764,$$

and 42 is the square root of 1764, being one of the two equal factors which compose it.

3. As above find the square root of each of the following: 11025, 32400, 245025, 145600, 48841.

4. What is the cube root of 74088?

As above resolving 74088 into its prime factors, we find
$2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \ 7 \cdot 7 \cdot 7$, *i. e.*, $2 \cdot 3 \cdot 7 \times 2 \cdot 3 \cdot 7 \times 2 \cdot 3 \cdot 7$, or $42 \times 42 \times 42$.
Hence $\sqrt[3]{74088} = 42$.

5. As above find the cube root of each of the following: 46656, 621875, 18399744, 4741632.

127. *Let the student write rules for extracting the square and cube roots, according to the above method.*

6. Find the 5th root of 5507317760000 by factoring, as above.

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* ELEMENTS OF ARITHMETIC (93).

GENERAL METHOD OF EXTRACTING ROOTS.

Of the Square Root.

128. The method already given for extracting roots gives a clear idea of the nature of a root, but is applicable only to perfect powers; it affords no method of approximating to the roots of numbers which are not exact powers of some number. Hence the necessity of the following more general methods.

[The ultimate aim in the following presentation is that the student learn to see in the formula $(x+y)^2 = x^2+2xy+y^2$, the rule for extracting the *Square Root;* in $(x+y)^3 = x^3+3x^2y+3xy^2+y^3$, the rule for the *Cube Root;* and, in general, to see the rule for the extraction of any root in the corresponding power of $x+y$.]

Ex. 1. Extract the square root of 7056.

SOLUTION.

$$\begin{array}{lrr}
(x+y)^2 = x^2+2xy+y^2 = & & \dot{7}05\dot{6}\,(84 \\
\quad x^2 \qquad\qquad\;\; = & & \underline{64\quad} \\
\qquad 2xy+y^2 = (2x+y)\,y. \quad \text{Now } 2x = & 160\,) & 656 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad y = & \underline{4} & \underline{656} \\
\qquad \text{Whence } 2x+y = \text{True Divisor,} & 164 &
\end{array}$$

EXPLANATION.—Pointing off the number 7056 by (***94***), we find that there will be two figures in the root, if 7056 is a perfect power; *i. e.*, the root will consist of a certain number of tens + a certain number of units. Let x represent the tens, and y the units, whence $x+y$ will represent the root,* and $(x+y)^2 = x^2+2xy+y^2$ will represent 7056. Now the square of the tens is the greatest square in the left-hand period, *i. e.*, in 70 (***94***); hence the tens digit is 8, whose

* Observe that $x+y$ represents the root only on the supposition that x represents *tens*. Thus, in this case, the root being 84, x is 80, not 8. If x represented *the tens digit*, the number would be represented thus, $10x+y$, *i. e.*, $(10\times 8)+4$.

square is 64. This 8 being tens its square is 6400, which subtracted from 7056 leaves 656. Hence $2xy+y^2$, which equals $(2x+y)y$, represents this remainder 656. But $2x$ is 2×80, or 160; and as $(160+y)y=656$, we can find y, approximately, at least, by dividing 656 by 160 as though $160y=656$. In this way we find that it is *probably* 4. If y *is* 4, $2x+y=160+4$, or 164, and $164y=656$. Multiplying 164 by 4 we find the product exactly 656. Hence 84 is the square root of 7056.

2. Extract the square root of 71690089.

OPERATION.

$$
\begin{array}{r|l}
 & \dot{7}1\dot{6}9\dot{0}0\dot{8}9\ (8467 \\
 & 64 \\
\hline
160 & 769 \\
4 & \\
\hline
164 & 656 \\
\hline
1680 & 11300 \\
6 & \\
\hline
1686 & 10116 \\
\hline
16920 & 118489 \\
7 & \\
\hline
16927 & 118489
\end{array}
$$

EXPLANATION.—Having found by pointing off that the root will consist of four figures, and that the highest order is 8, we observe that the square of the first two left-hand figures of the root, whatever they may be, will fall entirely in the two left-hand periods, since these two figures represent *hundreds*. Hence we proceed exactly as though we were extracting the root of 7169, or as in the preceding examples. Having found these two figures we may consider x in the formula $(x+y)^2=x^2+2xy+y^2$ as representing 84 *tens*, and y, the next figure in the root, exactly as though we were extracting the root of 716900, since the three left-hand figures of the root represent tens (846 tens), and the square of tens forms no part of the right-hand period. Then $2xy+y^2=(2x+y)y=(1680+y)y=11300$, approximately, and calling 1680 the trial divisor, we find that 6 is *probably* the next figure in the root. Now correcting the

divisor by adding this 6, $2x+y = 1680+6 = 1686$, which multiplied by 6 gives 10116.

In exactly the same manner letting the x in $(x+y)^2 = x^2+2xy+y^2$ represent 846 tens, we have taken out of the given number (71690089) the x^2, *i. e.*, (846 tens)^2, hence the remainder 118489 contains $(2x+y)y$, or $(16920+y)y$, and 16920 becomes the *Trial Divisor*.

REMARK 1.—It will be instructive to notice that the first subtrahend 64 is the square of the 8 thousands; the sum of the first two, 7056, the square of the 84 hundreds; the sum of the first three, 716716, the square of the 846 tens; and thus the sum of all the subtrahends, the square of the root found.

REMARK 2.—*The* x *of the formula always represents the part of the root already determined, and the* y *the next figure. Again, we never have occasion to consider the periods below the square of the figure we are seeking, since the square of this first part of the root can form no part of these lower periods.*

3. Extract the square root of 763.

$$
\begin{array}{r|l}
 & \dot{7}6\dot{3}\ (27.62+ \\
 & 4 \\
\hline
47 & 363 \\
 & 329 \\
\hline
546 & 34.00 \\
 & 32\ 76 \\
\hline
5522 & 1\ 2400 \\
 & 1\ 1044 \\
\hline
 & \quad 1356
\end{array}
$$

Having found the first figure in the root, 2, in seeking the second we have only to regard the two periods, since the square of this part of the root can not fall below these, *i. e.* can not be a decimal. There being a remainder after the square of 27 has been subtracted, we annex *two* 0's, since the squaring any decimal gives rise to twice as many decimal places in the power as there are in the root.* We now proceed as before, considering the x of the formula $(x+y)^2 = x^2+2xy+y^2$ as 27, and this y as the tenths sought, so that 34.00 contains $(2x+y)y$, *i. e.*, $(540+y)y$, or for trial $540y$, y being tenths and the 540 tenths. In like manner, and for a like reason, we annex the next two 0's, and proceed thus as far as may be desired.

REMARK 3.—When we commence annexing 0's the work can never terminate, since no digit multiplied by itself gives 0.

* This is evident, since the product contains as many decimal places as both factors.

4. Extract the square root of 7893, 450034, 58271, 11, 115, .3, .443, 64.8, 37.62, .046, 1.6.

REMARK 4.—In pointing off a decimal, or the decimal part of a mixed number, the periods must always be full, *i. e.*, have two figures, a 0 being annexed if the number of decimals is odd. Why?

5. Extract the square root of $\frac{25}{81}$, $\frac{16}{121}$, $\frac{49}{169}$, $\frac{64}{225}$, $\frac{289}{784}$.

Since a fraction is squared by squaring its numerator and denominator, how is the square root of a fraction extracted?

6. Extract the square root of the following, first reducing the fraction to a decimal: $\frac{3}{7}$, $\frac{5}{6}$, $\frac{3}{8}$, $5\frac{1}{2}$, $7\frac{2}{3}$, $10\frac{5}{6}$.

7. Extract the square root of $\frac{3}{11}$, by first making its denominator a perfect square.

$$\frac{3}{11}=\frac{3\cdot 11}{11\cdot 11}=\frac{33}{11\cdot 11}; \text{ whence } \sqrt{\frac{3}{11}}=\frac{\sqrt{33}}{11}=\frac{5.74+}{11}=.52+.$$

8. As in the last, extract the square root of each of the following: $\frac{2}{7}$, $\frac{5}{8}$, $\frac{7}{24}$, $\frac{1}{12}$, $\frac{1}{3}$, $\frac{11}{13}$.

Does this method possess any advantage over the reduction of the fraction to a decimal in the first instance? Try both on the same example.

129. *Let the student write the general rule for extracting the square root, and give the demonstration.*

The points to be made in the demonstration are:

1. Why the pointing is thus done.

2. Why seek the highest order in the root first.

3. Why the greatest square in the left-hand period is the square of the highest order in the root.

4. Why we bring down but one period at a time.

5. Why we form the trial divisor as we do.

6. Why we add to the trial divisor the last root figure found.

7. How we proceed when the first two figures have been determined, and why this is like the preceding part of the process.

Of the Cube Root.

Ex. 1. Extract the cube root of 300763.

The ultimate *practical* end here is to see the rule for extracting the cube root in the formula

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 = x^3 + (3x^2 + 3xy + y^2)\,y.$$

$$\begin{array}{r|l}
 & 300763\,(\underline{67} \\
 & \underline{216} \\
10800 & 84763 \\
1260 & \\
\underline{49} & \\
12109 & \underline{84763}
\end{array}$$

Pointing off as in (*95*), we see by (*95*) that the cube of the first figure in the root is 216, and hence that 6 is this first figure. Now, letting the x in the formula represent the six tens (60), after we have taken out the x^3 (216 thousand) we have

$$(3x^2 + 3xy + y^2)\,y = 84763.$$

For trial putting $(3x^2)\,y = 84763$, as $x = 6$ tens we have $3\,(6 \text{ tens})^2\,y = 84763$, or $10800y = 84763$, approximately. Hence 10800 is the *Trial Divisor*, and 7 is the probable next figure in the root. But the *True Divisor* is $3x^2 + 3xy + y^2$, or $3xy + y^2$ more than 10800. Now $3xy = 3 \times 60 \times 7$, and $y^2 = 7^2$, whence the true divisor is $10800 + 1260 + 49$, or 12109. Multiplying this by 7 we find no remainder. Hence 67 is the exact cube root of 300763.

2. Extract the cube root of 12812904.

The following operation shows the adaptation of the formula $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 = x^3 + (3x^2 + 3xy + y^2)\,y$.

$$\begin{array}{r|l}
 & 12812904\,(234 \\
 & \underline{8} \\
3x^2 = 3\,(2 \text{ tens})^2 = 1200 & 4812 \\
3xy = 3\,(2 \text{ tens})\,3 = 180 & \\
y^2 = 3^2 \qquad\quad = \underline{9} & \\
3x^2 + 3xy + y^2 = 1389 & 4167 = (3x^2 + 3xy + y^2)\,y. \\
\hline
3x^2 = 3\,(23 \text{ tens})^2 = 158700 & 645904 \\
3xy = 3\,(23 \text{ tens})\,4 = 2760 & \\
y^2 = 4^2 \qquad\quad = \underline{16} & \\
3x^2 + 3xy + y^2 = 161476 & 645904 = (3x^2 + 3xy + y^2)\,y.
\end{array}$$

REMARK.—The *True Divisor* each time is $3x^2+3xy+y^2$, the x being that portion of the root already found (regarded as tens), and the y the next figure sought. But for a *Trial Divisor* by means of which we may obtain this next figure approximately, we may take $3x^2$, *inasmuch as y is small as compared with x.* It will be seen that if y is not small as compared with x this approximation will not be close. Thus if x is 1 (ten), and y is 9, the division by the *Trial Divisor* would not be likely to get the next figure. But the trial divisor will never give too small a figure, and if it gives too large a one the completion of the divisor and its multiplication by this last figure will give too large a subtrahend.

3. Extract the cube root of 224755712.

To Extract the Cube Root.

130. RULE.—1. *Point off the number into periods by placing a mark over units and over each third figure therefrom* (***95***). *If there are decimals in the number, make the periods full, if they are not so, by annexing 0's.*

2. *Take the cube root of the highest cube in the left-hand period as the first figure in the root, subtract this cube from the left-hand period and to the remainder annex the next period, forming a dividend.*

3. *Take 3 times the square of the root already found, regarded as tens, as a* TRIAL DIVISOR, *by which divide the new dividend, thus obtaining (approximately) the next figure in the root. Complete the divisor by adding to the trial divisor 3 times the product of the root already found, regarded as tens, into the new root figure, and the square of that figure. Multiply this* TRUE DIVISOR *by the last root figure, subtracting the product from the last dividend, and to the remainder annex the next period, thus forming a new dividend.*

4. *Repeat the process described in the last paragraph till all the periods in the number are used; and, if the number is not exhausted, annex full periods of 0's and proceed in like manner till as many decimals are obtained in the root as may be desired.*

REMARK 1.—If, at any time, the trial divisor is not contained in the dividend to be used, according to the 3d paragraph in the rule, annex a 0 to the root and also two zeros to the trial divisor, bring down the next period, and then divide.

REMARK 2.—When the work does not terminate with the last period of significant figures it will not terminate at all, and the number is a surd. This is evident, since the right-hand figure in any subtrahend arises from cubing the corresponding digit in the root, and the cube of no digit produces 0 in units place.

DEM.—The reasons for pointing off as we do are, *first*, that it shows where the cube of the highest order in the root is to be found (*95*); then where the cube of the two highest orders; then of the three, etc. Thus the cube of the highest order in the root is found in the left-hand period; the cube of the two highest orders in the two left-hand periods, etc.; and, *second*, as a consequence of this, that we never need bring down and consider but one period at a time, since the cube of the part of the root we are seeking does not form any part of lower periods.

The reason for taking as a *Trial Divisor* 3 times the square of the part of the root already found is, that we are seeking a binomial, one part of which is the root already found and the other part of which is the figure we are seeking, and that the cube of this binomial is the largest cube in the last period brought down and the preceding ones. Thus letting x (tens) represent the part of the root already found and y the figure sought, we have $[x\,(\text{tens})+y]^3 = (x\text{ tens})^3+3\,(x\text{ tens})^2y+3\,(x\text{ tens})\,y^2+y^3$ as the cube of this part of the root, which cube is the greatest cube in the periods of the number under consideration. But of this cube we have in the previous part of the work removed the $(x\text{ tens})^3$; hence the remainder of this highest cube is $3\,(x\text{ tens})^2y+3\,(x\text{ tens})\,y^2+y^3$, or $[3\,(x\text{ tens})^2+3\,(x\text{ tens})\,y+y^2]\,y$; and *for a trial* we treat this remainder as $3\,(x\text{ tens})^2y$. Whence the divisor becomes $3\,(x\text{ tens})^2$.

That the *True Divisor* is obtained by adding to the *Trial Divisor*

3 times the product of the part of the root already found (regarded as tens) by the figure of the root sought, and the square of this latter figure, is evident from the formula, $3\,(x \text{ tens})^2 + 3\,(x \text{ tens})\,y + y^2$, the whole of which represents the True Divisor, while the first term is the *Trial Divisor*.

4. Apply the above reasoning in connection with the rule in extracting the cube root of 57856432.1632.

Where does the cube of the highest order in the root lie? Where the cube of the two highest orders? Where of the three? Why do we point off the decimal part thus, $.16\dot{3}20\dot{0}$, filling out the lowest period?

5. Extract the cube root of 11; of 7.64; of 2.7; of 4856432761005; of 200; of 178600; of 3400002.75.

6. Extract the cube root of the following, on the principle that a fraction is involved by involving numerator and denominator separately: $\frac{27}{125}$, $\frac{343}{1331}$, $\frac{8}{2197}$, $\frac{53441}{224765712}$, $\frac{1}{8}$.

7. Extract the cube root of the following by first making the denominator a perfect cube: $\frac{3}{5}$, $\frac{5}{7}$, $\frac{1}{2}$, $\frac{6}{11}$.

$$\frac{5}{7}=\frac{5\cdot7\cdot7}{7\cdot7\cdot7}=\frac{245}{7\cdot7\cdot7}. \quad \text{Therefore } \sqrt[3]{\frac{5}{7}}=\sqrt[3]{\frac{245}{7.7.7}}=\frac{\sqrt[3]{245}}{7},$$

$$\text{or } \tfrac{1}{7}\sqrt[3]{245}=\tfrac{1}{7} \text{ of } 6.257+. \quad \text{or } .893+.$$

To Extract Higher Roots than the Square and the Cube.

[This topic is of little importance, save as it throws light on the general subject; but, in this respect, it will be found of much service.]

131. *The 4th root may be extracted by taking the square root of the square root, and the 6th root by taking the cube root of the square root.*

This is evident, since the 4th root of a number is one of the four equal factors into which it may be conceived to be resolved, and if we resolve the number into two equal factors (extract the square root), and then resolve one of these factors into two equal factors (that is, extract its square root), the whole number will have been resolved into four equal factors. [Let the student show in like manner that the square root of the cube root, or the cube root of the square root, is the 6th root.]

Ex. 1. Extract the 4th root of 279841; of 7643256; of 8.5; of $\frac{16}{625}$.

2. Extract the 6th root of 601692057; of 825437502; of 53.2; of $\frac{64}{16807}$.

132. *Any root may be extracted by the method suggested by the corresponding power of a binomial, in a manner altogether similar to that given in (**130**) for extracting the cube root.*

We will illustrate this with an example:

3. Extract the 5th root of 4678757435232.

FORMULA. $(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5 = x^5 + (5x^4 + 10x^3y + 10x^2y^2 + 5xy^3 + y^4)y.$

OPERATION.

	4678757435232 (342
$x^5 = (3)^5 =$	243
$5\,(x \text{ tens})^4 = 5\,(30)^4 = 4050000$	22487574
$10\,(x \text{ tens})^3y = 10\,(30)^3\,4 = 1080000$	
$10\,(x \text{ tens})^2y^2 = 10\,(30)^2\,4^2 = 144000$	
$5\,(x \text{ tens})\;y^3 = 5\,(30)\;4^3 = 9600$	
$y^4 = 4^4 = 256$	
$5x^4 + 10x^3y + 10x^2y^2 + 5xy^3 + y^4 = 5283856$	$21135424 = (5x^4 + 10x^3y + 10x^2y^2 + 5xy^3 + y^4)\,y$
$5\,(x \text{ tens})^4 = 5\,(340)^4 = 66816800000$	135215035232
$10\,(x \text{ tens})^3y = 10\,(340)^3\,2 = 786080000$	
$10\,(x \text{ tens})^2y^2 = 10\,(340)^2\,2^2 = 4624000$	
$5\,(x \text{ tens})\;y^3 = 5\,(340)\;2^3 = 13600$	
$y^4 = 2^4 = 16$	
$5x^4 + 10x^3y + 10x^2y^2 + 5xy^3 + y^4 = 67607517616$	$135215035232 = (5x^4 + 10x^3y + 10x^2y^2 + 5xy^3 + y^4)\,y.$

133. ***Let the student write out a rule for the extraction of the 5th root.***

4. Extract the 5th root of 226435267111943.

5. Raise $x+y$ to the 4th power, and from the result write a rule for the extraction of the 4th root. Then apply the rule to extract the 4th root of 304758098401.

SYNOPSIS.

- RESOLUTION OF NUMBERS.
  - SUBTRACTION.
    - DEFINITIONS.
    - ORDER OF DEVELOPMENT. { 1, 2.
    - THREE FUNDAMENTAL PRINCIPLES. } Applications to
      - *Simple Numbers.*
      - *Compound Numbers.*
      - *Fractions.*
    - DIFFERENCE. { In two senses.
    - LITERAL NOTATION. } Positive and Negative.
      - *Addition.*
      - *Subtraction.*
        - Two Principles.
        - General Rule.
        - Demonstration.
      - *Multiplication.* { Law of Signs.
    - PARENTHESIS.
      - General Use.
      - Preceded by − sign.
  - DIVISION.
    - DEFINITIONS.—Two ideas.
    - RELATION TO SUBTRACTION.—TO MULTIPLICATION.
    - ORDER OF DEVELOPMENT. { 1, 2, 3.
    - FIVE FUNDAMENTAL PRINCIPLES. } Applications to
      - *Simple Numbers.*
      - *Compound Numbers.*
      - *Fractions.*
        - Common.
        - Decimal.
    - CONSEQUENCES OF DEFINITION. { 1, 2, 3, 4.
    - LITERAL NOTATION.
  - EVOLUTION.
    - DEFINITIONS.
    - BY FACTORING.
    - GENERAL RULE FROM $(x+y)^2$, $(x+y)^3$, $(x+y)^4$, etc.
      - Square Root.—Demonstration.
      - Cube Root.—Demonstration.
      - Higher Roots.
        - *Of even degree.*
        - *Of odd degree.*

CHAPTER V.

PROPERTIES OF NUMBERS.

SECTION I.

PROOF OF THE FUNDAMENTAL OPERATIONS BY CASTING OUT THE NINES.

[It is a serious mistake to suppose that this is a mere theoretical curiosity, and of no practical importance. There is no better and more simple check which we can apply to test the accuracy of our additions, subtractions, multiplications, and divisions. The author uses it habitually, as do many others.]

134. PROP.—*The remainder arising from dividing any integral number by 9 is the same as that which arises from dividing the sum of its digits by 9.*

DEM.—Let a, b, c, and d represent the digits of a number, whence the number is represented by $1000a+100b+10c+d$. Now since 1000 is $999+1$; 100, $99+1$; and 10, $9+1$, we have $1000a+100b+10c+d=(999+1)a+(99+1)b+(9+1)c+d=999a+a+99b+b+9c+c+d$ $=\overbrace{999a+99b+9c}^{\text{1ST PART.}}+\overbrace{a+b+c+d}^{\text{2D PART.}}$. Now the first part of the number when put in this form is evidently divisible by 9, hence whatever remainder there may be when the whole number is divided by 9 will arise from dividing the second part by 9. But the second part is the sum of the digits.

It is evident that the same argument is applicable to any number.

Ex. 1. Illustrate the above by showing that the remain-

der arising from dividing 5487 by 9 is the same as arises from dividing the sum of its digits by 9.

$$5487 = 1000\times5+100\times4+10\times8+7 = (999+1)5+(99+1)4+(9+1)8 + 7 = 999\times5+5+99\times4+4+9\times8+8+7 = \overbrace{5\times999+4\times99+8\times9}^{\text{1ST PART.}} + \overbrace{5+4+8+7}^{\text{2D PART.}}.$$

Now 9 exactly divides the first part, hence whatever remainder there may be when the whole number is divided by 9 will arise from dividing the second part by 9. But this second part is the sum of the digits, and gives a remainder 6 when divided by 9. So also does the number 5487.

2. Find according to (*134*) the remainder which would arise from dividing each of the following numbers by 9: 7856437: 95407532; 564037982054; 1234658; 27; 8826; 10000; 200; 84.

135. The most convenient practical method of procedure is to add the digits in succession, dropping 9 as often as the sum amounts to this number. Thus to reject the 9's from 785402356, we say "11;* 2, 5, 7, 11; 2, 7, 15; 6, 13; 4." This shows that the sum of the digits makes 4 9's and 4 remainder. But as the remainder is all we desire the number of 9's need not be attended to. A little practice enables us to see at a glance the excess of 9's. Thus no attention need be paid to the 9's which occur in the number or to any pair of digits which make 9. In the above the 6 and 3, 5 and 4, 7 and 2 could be disregarded, leaving only 8 and 5 to be added.

136. Any device by which we may test the accuracy of an operation in arithmetic by some other operation is called a ***Proof*** of the work.

Proof of Addition by casting out the 9's.

137. Ex. 1. Add 78543, 7652, 9867, 53216, 85764, and prove the work by casting out the 9's.

* In this, 9 is dropped from the sum at each semicolon.

OPERATION.

$$
\begin{array}{rr}
78543 = & x\ 9\text{'s} + \ 0\ \textit{Rem.} \\
7652 = & y\ 9\text{'s} + \ 2\ \text{"} \\
9867 = & z\ 9\text{'s} + \ 3\ \text{"} \\
53216 = & v\ 9\text{'s} + \ 8\ \text{"} \\
85764 = & w\ 9\text{'s} + \ 3\ \text{"} \\
\hline
235042 = & (x+y+z+v+w)\ 9\text{'s} + 16\ \textit{Rem.} \\
= & (x+y+z+v+w+1)\ 9\text{'s} + \ 7\ \textit{Rem.}
\end{array}
$$

EXPLANATION.—The 78543 contains, or is equal to, a certain number of 9's + 0 remainder. Now as we do not care particularly how many 9's this is, we call it x 9's. Moreover, as the remainder is all we need, we find it by (*135*). So 7652 is equal to y* 9's + 2 remainder, etc. Consequently the sum of all of these numbers is equal to as many 9's as there are in all of the numbers + all the remainders, *i. e.*, to $(x+y+z+v+w)$ 9's + 16 remainder. But 16 makes 1 9 and 7 remainder. Hence we have the entire sum equal to $(x+y+z+v+w+1)$ 9's + 7 remainder.

PRACTICAL SUGGESTION.—In practice, when we wish to prove the correctness of our work by casting out the 9's, we find by (*135*) the excess in all the numbers added, carrying the remainders right along from one number to the other, and find the total excess. Then reject the 9's from the sum, and if these excesses agree the work is probably right. Thus, having cast out the 9's from 78543 we have no remainder to carry forward to the next number; but having cast them out of 7652 we have 2 to carry to the 9867. So 2 + 7 are 9 and to be dropped, 8 + 6 = 14, giving an excess of 5 to be carried forward to the 53216, etc.

2. Ascertain whether 327471 is the sum of 23456, 78901, 43458, 58899, 81502, and 20855, without adding.

What is the excess of 9's in the several numbers? What in 327471?

* We could find exactly how many x, y, z, etc., represent by dividing the corresponding number by 9; but we do not need to know.

138. REMARK.—*In casting out the 9's from any number when the result is exactly 9 drop it; but when the sum becomes two digits add these digits and proceed.* Thus in casting the 9's out of 7643764286, we have $8+6=14$, $4+1=5$, $5+2+4=11$, $1+1=2$, $2+6+7=15$, $1+5=6$, etc., since the remainder, after 9 is rejected from 14, is the sum of its digits, etc.

3. Tell which of the numbers below is the sum of 825276, 704394, 37783, 1697, 349435, 697678, without adding. *Sum,* 3627543, 2328265, 2616263, 3425641.

Of course there is a possibility of getting a false sum which shall have the same excess as the true, but the probability is so small as to make the test of accuracy a good one for practical purposes.

Solve the following, proving the results by casting out the 9's, always using the process suggested in the *Remark* (***138***):

4. 78026 + 34826 + 8276 + 54073 + 8176.

5. 32753 + 2689592 + 123789 + 264 + 52771.

6. 3754821 + 4865 + 5643 + 86429 + 999 + 8879.

Proof of Multiplication by casting out the 9's.

139. PROP.—*The excess of 9's in the product of two numbers is equal to the excess in the product of the excesses in the two factors.*

DEM.—Any number used as a multiplicand may be considered as a certain number, say x, of 9's + a certain excess, which we will call r. Hence any multiplicand may be represented by x 9's + r, or $9x+r$. In like manner letting y represent the number of 9's in the multiplier and r' (read *prime*) the excess, the multiplier may be repre-

sented by $9y+r'$. Multiplying $9x+r$ by $9y+r'$ we have for the product $81xy+9x(r+r')+rr'$. But the sum of the first two terms, $81xy+9x(r+r')$ is evidently divisible by 9. Hence any excess of 9's which there may be in the product arises from the excess in $r\ r'$, *i. e.*, from the product of the excesses in the factors.

$$\begin{array}{r} 9x+r \\ \underline{9y+r'} \\ 9xr' \qquad +rr' \\ \underline{81xy+9xr \qquad\qquad} \\ 81xy+9x(r+r')+rr' \end{array}$$

Ex. 1. Multiply 734268 by 643, and prove by casting out the 9's. Also illustrate the preceding demonstration by means of this example.

In casting out the 9's from the multiplicand we say "14; 5, 7, 11; 2, 5, 12; 3." So in the multiplier we say "7, 13; 4." The number after the semicolon in each case should be obtained *not by absolutely rejecting 9 from the preceding*, but by taking the sum of its digits (*138*). The excess of 9's in the factors is 3 in the multiplicand and 4 in the multiplier, $4\times 3=12$, and $1+2=3$. Hence by (*139*) the excess of 9's in the product is also 3. Is 462134324 the product?

2. Select the product of 46343 by 437 from the following: 20151891, 20231891, 22251891, 20251891, 20251791.

3. Perform the following and prove by (*139*): 78654×8537; 604285×34682; 615.37×1.5; $\$3482.48 \times 1.10$; $\$7964.12 \times .08$.

Observe that this method of proof applies equally well to decimals as to whole numbers.

---

140. To prove Subtraction *by casting out the 9's, take the sum of the excesses in the subtrahend and remainder; and, if this equals the excess in the minuend, the work is probably correct.*

Student give the reason for this and the following, and apply it to examples which he can supply.

141. To prove Division by casting out the 9's, to the excess of 9's in the product of the excesses in the divisor and in quotient add the excess in the remainder; and, if the excess in this sum equals the excess in the dividend, the work is probably correct. [Why?]

Show by casting out the 9's that the following are correct:

1. [illegible] ÷ [illegible] = [illegible], with remainder 503.
2. [illegible] ÷ [illegible] = [illegible], with remainder 414.
3. [illegible] ÷ [illegible] = [illegible].
4. [illegible] ÷ [illegible] = [illegible].
5. [illegible] ÷ 55 = [illegible].
6. [illegible] ÷ [illegible] = [illegible].

SECTION II.

TESTS OF DIVISIBILITY OF NUMBERS.

142. One number is said to be ***Divisible*** by another when the former contains the latter an integral number of times without a remainder.

143. Any number is divisible by 1.

This is self-evident.

144. Any number is *divisible by* 2, if the right-hand figure is 0, or a digit which is divisible by 2, and not otherwise.

DEM.—Any number may be considered to be as many 10's as are represented by the figures exclusive of the right-hand one, + the right-hand figure. Now the first part is divisible by 2, since 10 is so divisible. Hence if the 2d part is 0, or is divisible by 2, the whole number is, but not otherwise.

145. An Even Number is a number which is divisible by 2. ***An Odd Number*** is one which is not divisible by 2. By (***144***) **any number is even which ends with an even digit, or 0, and odd if it ends with an odd digit.**

146. **1. It is possible that a number ending in any figure is divisible by 3.**

2. Any number is divisible by 3, if the sum of its digits is so divisible, and not otherwise.

DEM.—1. There is no digit that may not be produced in the right-hand place by multiplication; thus $3 \times 7 = 21$, $3 \times 4 = 12$, $3 \times 1 = 3$, $3 \times 8 = 24$, $3 \times 5 = 15$, $3 \times 2 = 6$, $3 \times 9 = 27$, $3 \times 6 = 18$, $3 \times 3 = 9$.

2. a, b, c, and d being the digits of a number, we have $(3+7)^3a + (3+7)^2b + (3+7)c + d$ representing the number. Now, performing the operations indicated in any term, as $(3+7)^3a$,* each term of the result contains a certain number of 3's, + some power of 7 × by one of the digits. But all those terms consisting of a certain number of 3's are, of course, divisible by 3; hence we have to examine those consisting of powers of 7 × the respective digits. These are of the form $7^3a + 7^2b + 7c + d$, or $(6+1)^3a + (6+1)^2b + (6+1)c + d$. Performing the operations indicated in the latter forms, there arise terms containing 6 as a factor, and terms consisting of the single digits (see foot-note): the former are divisible by 3; hence, if the latter (the sum of the digits) is divisible by 3, the entire number is.

REMARK.—Any number divided by 3 gives the same remainder as the sum of its digits divided by 3. Hence the fundamental operations can be proved by casting out the 3's, in the same manner as by casting out the 9's.

---

* $(3+7)^3a = (3^3 + 3 \cdot 3^2 \cdot 7 + 3 \cdot 3 \cdot 7^2 + 7^3)\,a = (3^2 + 3^2 \cdot 7 + 3 \cdot 7^2)\,3a + 7^3a.$

$(3+7)^2b = (3^2 + 2 \cdot 3 \cdot 7 + 7^2)\,b = (3 + 2 \cdot 7)\,3b + 7^2b.$

$(3+7)\,c = 3c + 7c.$ Hence the number may be written:

$$\overbrace{(3^3 + 3^2 \cdot 7 + 3 \cdot 7^2)\,3a + (3 + 2 \cdot 7)\,3b + 3c}^{\text{1ST PART.}} + \overbrace{7^3a + 7^2b + 7c + d}^{\text{2D PART.}}.$$

In a similar manner the 2d part may be written:

$$(6+1)^3a + (6+1)^2b + (6+1)\,c + d = (6^3 + 3 \cdot 6^2 + 3 \cdot 6 + 1)\,a + (6^2 + 2 \cdot 6 + 1)\,b + (6+1)\,c + d$$
$$= \overline{(6^2 + 3 \cdot 6 + 3)\,6a + (6+2)\,6b + 6c} + \overline{a + b + c + d}.$$

147. Any number is divisible by 4 if the number represented by its last two digits is divisible by 4, or if the last two figures are 0's, and not otherwise.

DEM.—Any number is composed of as many hundreds as are represented by the figures, exclusive of the two right-hand ones, + the number represented by these figures. But any number of 100's is divisible by 4. Hence if the number represented by the two right-hand figures is divisible by 4, or if the last two figures are 0, the entire number is divisible by 4, and not otherwise.

148. Any number is divisible by 5 if the right-hand figure is 0, or 5, and not otherwise.

Let the student give demonstration.

149. Any even number the sum of whose digits is divisible by 3 is divisible by 6, and no other number is so divisible.

Let the student give demonstration.

REMARK.—No odd number is divisible by 6.

150. It is possible that a number ending with any figure is divisible by 7.

For method of proving this see (*146*).

There is no convenient practical test of divisibility by 7. From

$$10^3a+10^2b+10c+d=(7+3)^3a+(7+3)^2b+(7+3)c+d$$

$$=(7^2+3\cdot7\cdot3+3\cdot3^2)7a+(7+2\cdot3)7b+7c+\overline{3^3a+3^2b+3c+d},$$

the student may observe that to be divisible by 7, the units digit + 3 times the tens + 3^2 times the hundreds + 3^3 times the thousands, must be divisible by 7. Also, from $3^3a+3^2b+3c+d$, making b and c each 0, and $d=a$, and observing that $3^3a+a=27a+a=(4\cdot7-1)a+a=4\cdot7a-a+a=4\cdot7$, he may observe that any number consisting of 4 places, with the same digit in the 1st and 4th and the intermediate figures 0, is divisible by 7.

151. Any number is divisible by 8 if the last three figures are 0, or if the number represented by them is divisible by 8, and not otherwise.

Demonstration similar to (*147*); let the student give it.

152. Any number is divisible by 9 if the sum of its digits is so divisible, and not otherwise.

For demonstration see (***134***). Let the student adapt that demonstration to this statement.

153. Any number is divisible by 10 when the right-hand figure is 0, and not otherwise.

Let the student give the reason.

154. In order to be divisible by a composite number, a number must be divisible by all the factors of the composite number.

This is evident from the principle in Division, in accordance with which we may divide by any composite number by dividing by its factors in succession (***115***).

155. Any number is divisible by 11 if the difference between the sum of the digits in the odd places, counting from units, and the sum of those in the even places is 0, or is divisible by 11, and not otherwise.

DEM.—The digits being a, b, c, d, and e, we have, as before, the number $(11-1)^4a+(11-1)^3b+(11-1)^2c+(11-1)d+e$. Now upon expanding the terms, we find that all the terms contain 11 as a factor except the following, $a-b+c-d+e$. Hence if this part is 0, or is divisible by 11, the number itself is.

REMARK.—From this it follows that any number represented by an even number of figures, with the same digit in the extreme places and 0's in the intermediate places, as 5005, 700007, 30000003, etc., is divisible by 11.

156. Any number is divisible by 12 if the number represented by the two right-hand digits is divisible by 4, or is 0, and the sum of all the digits is divisible by 3.

This will be seen to be a consequence of (***154***), in connection with (***144, 146***).

157. It is possible that a number ending in any digit is divisible by 13. Again, any number represented by 4 figures, the two extremes being the same digit, and the intermediate figures 0's, is divisible by 13.

The student will be able to discern the reason for the first statement, and the second appears from the following:

$$(13-3)^3a+(13-3)^2b+(13-3)c+d;$$
$$-3^3a+3^2b-3c+d.$$

Putting $b=0$, $c=0$, and $d=a$,

$$-27a+a=-26a,$$

which is divisible by 13.

But we have carried the subject far enough for the student to discover the spirit and method of such investigations, and must now dismiss it.

Ex. 1. Write 5 numbers, each divisible by 2. Each by 3. 4. 5. 6. 7. 8. 9. 10. 11. 12. 13. Let each number be represented by 4 or more figures.

2. By what numbers less than 13 is 564256 divisible. Determine by the above test, without actually dividing.

3. By what numbers between 1 and 15 inclusive is 360360 divisible?

SECTION III.

COMMON DIVISORS.

158. A Divisor of a number is an integer* which divides it without a remainder. A divisor of a number is therefore a factor. A *Divisor* is also called a *Measure.*

* In arithmetic it is expedient to restrict these definitions thus; but in the extension of the notions to the literal notation this limitation is soon lost sight of.

159. A Common Divisor of two or more numbers is a common integral factor; that is, a whole number which exactly divides each of the numbers. A common divisor is therefore a common factor.

160. The Greatest Common Divisor of two or more numbers is the greatest whole number which will exactly divide each of them.

161. *One* is the least divisor of a number, and the number itself is the greatest.

162. A Composite Number is a number which is composed of integral factors other than itself and unity.

163. A Prime Number is a number which has no integral factor other than itself and unity. Numbers are said to be prime to each other when they have no common factor.

164. PROP. 1.—*A number which contains a factor not in a given number will not divide that given number.*

This is a direct consequence of the principle in division by which we divide a number by dividing successively by its factors (***115***); since we should thus have to divide the number by a factor not contained in it; which is a contradiction in terms.

165. PROP. 2.—*A number is divisible by the product of any number of its prime factors, no factor being used more times than it occurs in the number.*

Thus a, b, c, and d being the prime factors of a number the product of a and b is contained $c \times d$ times in the number, since $a \times b \times c \times d$ composes the number. So $a \times b \times c$ is contained d times, etc.

166. Prop. 3.—*The Greatest Common Divisor of two or more numbers is the product of all their common prime factors.*

By (Prop. 2) such a number divides each of the proposed numbers; and by (Prop. 1) if any other factor were introduced it would not be a divisor of the number which does not contain that factor.

Ex. 1. Find the G. C. D.* of 1512, 882, and 630.

$1512 = 2 \times 2 \times 2 \times 3 \times 3 \times 3 \times 7$, $882 = 2 \times 3 \times 3 \times 7 \times 7$, and $630 = 2 \times 3 \times 3 \times 5 \times 7$. The prime factors common to all are 2, 3, 3, 7. Hence $2 \times 3 \times 3 \times 7 = 126$ is the G. C. D. of 1512, 882, and 630.

167. *Let the student write a rule for finding the G.C.D. according to the principles now developed.*

2. Find the G. C. D. of 3252 and 4248. Of 506 and 308. Of 110, 140, and 680. Of 56, 84, 140, and 168.

General Method of finding the G. C. D.

168. For cases in which it is difficult to determine the factors of the numbers, as well as for the purposes of the Higher Mathematics, a rule based upon the two following propositions is important.

169. Prop. 1.—*A divisor of a number is a divisor of any number of times that number.*

This is self-evident, since if a is contained in b, c times, it is contained in $2b$ twice as many times, in $3b$ 3 times as many times, etc.

170. Prop. 2.—*A divisor of any two numbers is a divisor of their sum and also of their difference.*

* Abbreviation for Greatest Common Divisor.

This also is self-evident. Thus if a is contained in m 5 times, and in n 3 times, it is evident that it is contained in their sum 8 times, and in their difference 2 times.

171. Rule.—1. *To find the G. C. D. of two numbers, divide the greater by the less, and this divisor by the remainder, continuing to divide the last divisor by the last remainder until there is no remainder. The last divisor is the G. C. D. sought.*

$$\begin{array}{l} 42\,)\,138\,(\,3 \\ \qquad\;\underline{126} \\ \qquad\;\;12\,)\,42\,(\,3 \\ \qquad\qquad\;\;\;\underline{36} \\ \qquad\qquad\;\;\;\;6\,)\,12\,(\,2 \\ \qquad\qquad\qquad\quad\;\underline{12} \end{array}$$

Dem.—In order to demonstrate this rule, let us find the G. C. D. of 42 and 138. Performing the operation according to the rule, as in the margin, we are to prove that 6 is the G. C. D. of 42 and 138.

As 42 is its own G. D., if it divides 138 it is the G. C. D. sought. Trying it, we find a remainder 12. Now any divisor of 42 is a divisor of 3 times 42, or 126 (Prop. 1), and any divisor of 126 and 138 is a divisor of their difference 12 (Prop. 2). Hence the G. C. D. sought cannot be greater than 12. Moreover any number which divides 12 and 42 divides 138 which is the sum of 12 and 3 times 42 (Props. 1 and 2). Thus the question is reduced to finding the G. C. D. of 12 and 42.

In like manner we can reduce it to the question of finding the G. C. D. of 6 and 12. But this is 6. Hence 6 is the G. C. D. of 42 and 138.

Ex. 1. Find the G. C. D. of 9131 and 13133.

OPERATION.

13133	9131	1
9131	8004	2
4002	1127	3
3381	621	1
621	506	1
506	460	4
115	46	2
92	46	2
23		

An arrangement like that in the margin will be found convenient in performing the divisions. Placing the first divisor, the smaller number, on the right, divide and write the quotient at the right of both. Thus 9131 is contained in 13133, 1 time, with 4002. Now using this remainder as divisor,

divide 9131 just as it stands, writing the quotient underneath the former. Thus 4002 is contained in 9131, 2 times with 1127 remainder. Now 1127 becomes the divisor and 4002 the dividend and the same order is repeated.

The student should give the argument, as in the demonstration above, in connection with a sufficient number of examples to make it perfectly familiar. In this example the first step is to show that the G. C. D of 9131 and 13133 is also the G. C. D, of 4002 and 9131. This consists in showing that the G. C. D. sought cannot be greater than 4002, and that any divisor of 4002 and 9131 is also a divisor of 9131 and 13133. This being clearly seen, the same argument repeated shows that the G. C. D. sought is the G. C. D. of 1127 and 4002; then of 621 and 1127; and finally of 23 and 46.

2. Find the G. C. D. of 156479 and 100259. Of 3252 and 4248. Of 1825 and 2555.

When there is an evident common factor, as 5 in this case, it is expedient to divide by it, and operate on the quotients. The common factor or factors thus reserved are to be multiplied into the G. C. D. of the numbers after these were rejected. Student give the reason.

2555	1825	5*
511	365	1
365	292	2
146	73	2
146		

$73 \times 5 = 365 =$ G. C. D.

3. Find the G. C. D. of 506 and 308. Of 556 and 672.

Whenever in the course of the process a common factor appears, divide it out and reserve it as a factor of the G. C. D. Moreover, whenever it be-

506	308	2*
253	154	1
154	99	1
99	55	11*

$2 \times 11 =$ G. C. D.

672	556	2*
336	278	2*
168	139	1
139	116	4
29	23	

$2 \times 2 =$ G. C. D.

comes evident that the two numbers to be compared are prime to each other, the work may cease.*

---

4. Find the G. C. D. of 4420, 3094, and 1326. Of 1445, 1190, and 204.

The common method of solving such problems is to find the G. C. D. of the two least numbers, and then of this G. C. D. and the next larger of the numbers, etc. But familiarity with the principles on which the operations are based will suggest better ways. Thus, for the above we have the following:

4420	3094	1326	2*
2210	1547	663	2, 3.
2210	1326	663	10
	221		
	2		
	442 = G. C. D.		

1445	1190	204	5, 8.
1360	1020	170	2,
85	170	34	2, 5, 2.
68	170	34	
17 = G. C. D.			

In the *first* of these we reserve the evident common factor 2 as a factor in the G. C. D. sought. Hence we have to find the G. C. D. of 663, 1547, and 2210. Dividing 1547 by 663, we reduce the problem to finding the G. C. D. of 663, 221, and 2210. Then as 221 is found to be the G. C. D. of these, $221 \times 2 = 442 =$ G. C. D. sought.

In the second, the problem is first reduced to finding the G. C. D. of 204, 170, 85; then as 85 is found to be a divisor of 170, it becomes a question between 85 and 204. The order of operation in this case is, $1190 \div 204$; $1445 \div 170$; $170 \div 85$; $204 \div 85$; $85 \div 34$; $34 \div 17$.

The student will have no difficulty in applying this method of finding the G. C. D. of several numbers, if he is careful to mark at each step what numbers are now to be examined, and always divides first by the least number, proceeding in order to the greatest under comparison.

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* These principles become especially important when this process is applied to literal quantities.

5. Find the G. C. D. of 805, 1978, and 1311. Of 1134, 837, and 1347. Of 2113, 3391, and 1481. Of 1554, 1998, 2220, and 2886.

2886	2220	1998	2*
1443	1110	999	1, 13.
111	999	999	9
333	111		
333			

$111 \times 2 = 222 =$ G. C. D.

We conclude this section with the following, giving it rather as a curiosity and matter of historic interest than for careful study.

172. To find the prime numbers between any given limits.—*Write down all the odd numbers, 1, 3, 5, 7, 9, etc. Over every third from 3 write 3; over every fifth from 5 write 5; over every seventh from 7 write 7; over every eleventh from 11 write 11; and so on. Then all the numbers which are thus marked are composite; and the others, together with 2, are prime.* (Why?)

Also the figures thus placed over, are the factors of the numbers over which they stand. (Why?)

Ex. Find all the prime numbers less than 100.

$1 \quad 3 \quad 5 \quad 7 \quad \overset{3}{9} \quad 11 \quad 13 \quad \overset{3\cdot5}{15} \quad 17$

$19 \quad \overset{3\cdot7}{21} \quad 23 \quad \overset{5}{25} \quad \overset{3}{27} \quad 29 \quad 31 \quad \overset{3\cdot11}{33} \quad \overset{5\cdot7}{35}$

37	$\overset{3\cdot 13}{39}$	41	43	$\overset{3\cdot 5}{45}$	47	$\overset{7}{49}$	$\overset{3\cdot 17}{51}$	53
$\overset{5\cdot 11}{55}$	$\overset{3\cdot 19}{57}$	59	61	$\overset{3\cdot 7}{63}$	$\overset{5\cdot 13}{65}$	67	$\overset{3\cdot 23}{69}$	71
73	$\overset{3\cdot 5}{75}$	$\overset{7\cdot 11}{77}$	79	$\overset{3}{81}$	83	$\overset{5\cdot 17}{85}$	$\overset{3\cdot 29}{87}$	89
$\overset{7\cdot 13}{91}$	$\overset{3\cdot 31}{93}$	$\overset{5\cdot 19}{95}$	97	$\overset{3\cdot 11}{99}$				

Hence, rejecting all the numbers which have *superiors*, the primes less than 100 are 1, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, together with the number 2.

This process may be extended indefinitely, and is the method by which primes are found even by modern computers. It was invented by Eratosthenes, a learned librarian at Alexandria (born B. C. 275). He inscribed the series of odd numbers upon parchment; then cutting out such numbers as he found to be composite, his parchment with its holes somewhat resembled a *sieve*; hence, this method is called "*Eratosthenes' Sieve.*"—[*Sangster's Arithmetic.*]

SECTION IV.

MULTIPLES.

173. ***A Multiple*** of a number is a number which contains that number as a factor. Hence a *Composite Number* is a multiple of each of its factors.

174. ***A Common Multiple*** of two or more numbers is a multiple of each of them.

175. The Least Common Multiple of two or more numbers is the least number which is a multiple of each of them. Hence the L. C. M. of several numbers cannot be less than the greatest number.

176. The Product of two or more numbers is a *Common Multiple* of them all, since it contains each of them as a factor.

The student should give examples illustrating the above definitions.

177. A Multiple of a number must contain all its factors.

Thus to say that 105 is a multiple of 35, is to say that it is divisible by 35, *i. e.*, by 5 and 7, the factors of 35.

Ex. 1. Find a C. M. of 5, 7, 6, and 3. (***176.***) How many common multiples of these numbers can you find? What is the L. C. M. of them?

2. Find the L. C. M. of 11 and 13. Of 17, 23, and 5.

3. Find the L. C. M. of 12, 60, 84, and 132.

Since the L. C. M. of these numbers must contain the greatest of them (***175***), it cannot be less than 132, the factors of which are 2, 2, 3, and 11. Now, as the factors of 84 are 2, 2, 3, and 7, 132 does not contain 84, since it lacks the factor 7. But as it has the other factors of 84, if we introduce the 7 the product will contain 84. Hence 132×7 is the L. C. M. of 132 and 84. In like manner, $60 = 2 \cdot 2 \cdot 3 \cdot 5$, of which factors 132×7 contains all but the 5. Introducing this, we have $132 \times 7 \times 5$ as the L. C. M. of 132, 84, and 60. Finally, as the factors of 12 are contained in this product, $132 \times 7 \times 5 = 4620$ is the L. C. M. of 12, 60, 84, and 132.

We have $4620 = 2 \cdot 2 \cdot 3 \cdot 11 \cdot 7 \cdot 5$. If one of the factors 2 be stricken out, which of the numbers will it cease to be a multiple of? If the factor 3? Why? If the factor 7? Why? If 5? Why? If 11?

178. *Let the student write a rule for this method of finding the L. C. M. of two or more numbers.*

4. Find the L. C. M. of 16, 24, 32, and 40.

OPERATION.

$$40 = 2 \cdot 2 \cdot 2 \cdot 5\ ;$$

$$32 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2\ ;\quad 24 = 2 \cdot 2 \cdot 2 \cdot 3\ ;\quad \text{and}\quad 16 = 2 \cdot 2 \cdot 2 \cdot 2.$$

We see that 40 lacks two factors 2, which 32 has, and a factor 3 which 24 has. Introducing these, the product will contain all the factors of any one of the numbers, and no more. Hence $40 \cdot 2 \cdot 2 \cdot 3 = 480$ is the L. C. M. of the numbers 16, 24, 32, and 40.

5. Find the L. C. M. of 14, 28, 21, and 42.
6. Find the L. C. M. of 40, 60, 90, 80, and 30.

Find the L. C. M. of the following:

7. 27, 54, 81, 14, 63.
8. 15, 42, 72, 81.
9. 45, 81, 96, 35.
10. 60, 50, 144, 35, 18.
11. 14, 21, 3, 2, 63.
12. 41, 37, 11, 23, 71.
13. 2, 6, 48, 36, 24.
14. 20, 60, 15, 165, 210, 63.

179. The following rule is often given for finding the L. C. M. of several numbers. Let the student show that the principle involved is the same as that explained above. Write a formal demonstration of the rule.

I. *Write the numbers in a horizontal line, and divide by any prime number that will divide two or more of them without a remainder, placing the quotients and numbers undivided in a line below.*

II. *Divide this line as before, and thus proceed till no two numbers are divisible by any number greater than 1. The continued product of the divisors and numbers in the last line will be the L. C. M. of the numbers.*

$$\begin{array}{r|rrrr} 5 & 45 & 81 & 96 & 35 \\ \hline 9 & 9 & 81 & 96 & 7 \\ \hline 3 & 1 & 9 & 96 & 7 \\ \hline & 1 & 3 & 32 & 7 \end{array}$$

$5\cdot 9\cdot 3\cdot 3\cdot 32\cdot 7 = \text{L. C. M.}$

Applying this rule to the solution of the 9th example above, we have the work in the margin.

It may be well for the student to solve the examples above by this method.

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15. Find the L. C. M. of 54479, and 35741.

In this case it is not easy to discern a common factor, if the numbers have one. We may therefore apply the method for finding the G. C. D. (*171*). Having found the G. C. D. we can divide the smaller number by it and find the factor by which the larger number is to be multiplied in order to give a product which will contain the smaller.

16. Find the L. C. M. of 31861, 88409, and 63269.

SYNOPSIS.

PROPERTIES OF NUMBERS.
- OF 9.—PROOF OF FUNDAMENTAL RULES BY.
- TESTS OF DIVISIBILITY.
- COMMON DIVISOR. { G. C. D. { *By Factoring.* / *General Rule.* { Principles. / Demonstration.
- MULTIPLES. { Common Multiples. / L. C. M. { *To find.* *Rule.* *Dem.*

CHAPTER VI.

COMPARISON OF NUMBERS.*

SECTION I.

SIMPLE EQUATIONS.

180. *An Equation* is an expression in mathematical symbols, of equality between two numbers or sets of numbers.

181. *The First Member* of an equation is the part on the left hand of the sign of equality. ***The Second Member*** is the part on the right.

Thus $4x=20$, $3x-5=2x+4$, are equations. In the first, $4x$ is the *first member*, and 20 is the second member. In the second, $3x-5$ is the first member, and $2x+4$ is the second member.

182. In solving problems by means of equations, we commonly use x, y, z, or some letter in the latter part of the alphabet to stand for the answer, which we call the ***Unknown Quantity.*** The numbers which are mentioned in the problem, as those which we are to use in

* This subject, the Comparison of Numbers, is the proper subject of Algebra, and hence we have not included it in our synopsis of Pure Arithmetic. The merest elements are presented here, primarily for their utility in the solution of the practical business problems of Arithmetic, but also as an introduction to the subject of Algebra proper.

obtaining the answer, are called ***Known Quantities,*** and are represented by figures, or by the letters a, b, c, d, etc., in the first part of the alphabet.

Thus, suppose we have the problem, "What number is that from which if you subtract 12, and then add twice the number itself, the sum will be the number itself minus 4?" The number we seek, *i. e.*, the answer, we represent by x, and call it the *Unknown Quantity.* The other numbers, which we call the *Known Quantities*, are given in figures in this problem.

To make an equation out of this problem, we consider that it says, "If you subtract 12 from the number (which will be $x-12$), and then add twice the number (which will be $x-12+2x$), the sum will be the number itself (x) minus 4 (that is, $x-4$)." Hence the equation is $x-12+2x=x-4$.

Ex. 1. Express the following in an equation: "Said a gentleman to me, if with the money I now have, I had twice as much, and \$14 more, I should have \$74. How much money had he?"

The *Unknown Quantity* is the money he had. We therefore call this x, and twice as much is $2x$. Thus $2x+14$ is 14 more than twice as much money as he had. But this he said was \$74. Hence the equation is $2x+14=74$.

What is the *first member?* What the *second* member? What are the *known quantities?*

2. Express the following in an equation: "Said a gentleman to me, if with the money I now have, I had a times as much, and b dollars more, I should have m dollars. How much money had he?

As before, x is the unknown quantity, a times as much is ax, and b dollars more is $ax+b$. But this is said to be m dollars; so that we have $ax+b=m$, as the equation.

183. This expressing the conditions of a problem in the form of an equation is called ***Stating*** the problem, or making the ***Statement.***

State* the following:

3. A farmer sold 13 bushels of barley at a certain price, and afterwards 17 bushels more at the same price, at the second time receiving 36 shillings more than at the first. What was the price of a bushel?

4. A farmer sold a bushels of barley at a certain price, and afterwards b bushels more at the same price, at the second time receiving c shillings more than at the first. What was the price of a bushel?

In these two examples, what is the unknown quantity? 13 bushels at x shillings per bushel amount to how much? a bushels at x shillings per bushel is how much? How do you represent the difference between $17x$ and $13x$? What is this difference said to be? How do you represent the difference between bx and ax? What is this said to be?

5. A gentleman once had 40 dollars; he spent a certain part of it, and found that he had 3 times as much left as he had spent. How much money had he spent?

6. A gentleman once had a dollars; he spent a certain part of it, and found that he had b times as much left as he had spent. How much money had he spent?

7. A gentleman being asked the age of his son, answered, "My wife is 23 years older than my son, I am 10 years older than my wife, and the sum of our ages is 110 years." What was the age of the son?

Son's age, x, the *unknown quantity.*
Wife's age, $x+23$; man's age, $x+23+10$.
The equation is $x+x+23+x+23+10=110$.

* Do not attempt anything more at present. Our purpose is to develop the idea of an equation. The solution will be attended to presently.

8. A gentleman being asked the age of his son, answered, "My wife is c years older than my son, I am m years older than my wife, and the sum of our ages is n years." What was the age of the son?

Son's age, x, the *unknown quantity.*
Wife's age, $x+c$; man's age, $x+c+m$.
The equation is $x+x+c+x+c+m=n$.

9. A gentleman, being asked how much money he had in his pocket, answered that the fourth and fifth part amounted together to 9 dollars. How much money had he?

Equation, $\frac{x}{4}+\frac{x}{5}=9$.

10. A gentleman, being asked how much money he had in his pocket, answered that the a^{th} part and b^{th} part amounted to m dollars. How much money had he?

Equation, $\frac{x}{a}+\frac{x}{b}=m$.

11. To find a number such that if one-third of this number be added to itself the sum will be 60.

12. To find a number such that if one a^{th} of this number be added to itself the sum will be b.

Transformations of Equations.

184. After having stated a problem, as explained above, the next thing is to find from the equation what the value of the unknown quantity is; that is, *find the answer.* The equation is only an instrument for solving the problem.

185. The process by which we find the value of an unknown quantity involved in an equation is this: *We make such changes as will leave the unknown quantity alone on one member of the equation, and all the known quantities on the other.* These changes are called ***Transformations;*** but in making them *it is essential that we do nothing which will destroy the equality of the members.*

Ex. 1. What changes can you make upon the equation $3x - 5 = 2x + 6$ which will not destroy the equality of the members, and which will finally leave x alone in the first member?

Were you to take the -5 from the first member, would it make that member less, or greater? What then must be done to the second member to preserve the equality of the members? Then, if $3x-5=2x+6$ is a true equation, is $3x=2x+11$ true also? Why?

Again, if $2x$ is taken away from the second member of $3x=2x+11$, does the second member become greater, or less? What then must be done to the first member to preserve the equality of the members? If $3x-2x=11$, does it follow that $x=11$? Why?

There are other changes which can be made on the members of $3x-5=2x+6$, without destroying the equality of the members, but those suggested above are such as are necessary in order to find the value of x. Is $3x-5=2x+6$ when $x=11$? What is $3x-5$ when $x=11$? What is $2x+6$ when $x=11$? Then when $x=11$, is $3x-5=2x+6$?

Is $3x-5=2x+6$ if $x=3$? If $x=7$? Can you find any number other than 11, which substituted for x makes $3x-5=2x+6$?

2. If $\frac{x}{3}+4=9-\frac{x}{2}$, what is the value of x?

Will it destroy the equality of the members if we multiply each member by 6? Why not? What is 6 times $\left(\frac{x}{3}+4\right)$? What is 6 times $\left(9-\frac{x}{2}\right)$? Then if $\frac{x}{8}+4=9-x$, is $2x+24=54-3x$? Why?

How do you multiply a fraction by a number equal to its denominator, as $\frac{x}{3}$ by 3? Now how can we get the $-3x$ out of the second member and not destroy the equation? Is $2x+24+3x=54$, a true equation deduced from $2x+24=54-3x$? Why? Then if $2x+24+3x=54$, does $2x+3x=30$? Why? If $5x=30$, what does $x=$?

186. The process of changing a term from one member of an equation to the other without destroying the equality of the members is called ***Transposition.***

Rule.*—To transpose a term, drop it from the member in which it stands and insert it in the other member with the sign changed.*

187. A figure or letter written before another to tell how many times it is to be taken is called a ***Coefficient.***

Thus in $5x$, 5 is the coefficient of x. So in ax, a is the coefficient of x. In $3bx$, $3b$ is the coefficient of x.

188. *There are* ***four principal*** *transformations of simple equations containing one unknown quantity, viz.:* ***Clearing of Fractions, Transposition, Collecting Terms,*** *and* ***Dividing by the coefficient of the unknown quantity.***

189. These transformations are all based on the two following *axioms*:

AXIOM 1. *Any operation may be performed upon any term or upon either member, which does not affect the value of that term or member, without destroying the Equation.*

AXIOM 2. *If both members of an Equation are increased or diminished alike, the equality is not destroyed.*

3. If $\frac{4x}{5} + 6 = \frac{2x}{3} + 12$, what is the value of x?

If $\frac{4x}{5}+6=\frac{2x}{3}+12$, does $12x+90=10x+180$? Why?

If $12x+90=10x+180$, does $2x=90$? Why?

If $2x=90$, does $x=45$? Why?

If you put 45 for x in the equation in the example, are the members equal? Can you find any other number which substituted for x will make the members equal? Try 15, 90, 60.

190. General Rule for the Solution of a Simple Equation.

1. *If there are fractions, clear the equation of them by multiplying each member by the L. C. M. of all the denominators.*

2. *Transpose the terms containing the unknown quantity to the first member, if they are not already there, and those not containing the unknown quantity (the known terms) to the second member, if they are not already there.*

3. *Unite all the terms containing the unknown quantity into one term, and put the terms in the second member into the simplest form.*

4. *Divide each member by the coefficient of the unknown quantity.*

4. Given $7x - 16 = 3x - 4$ to find x.
5. Given $4x - 48 = 24 - 2x$ to find x.
6. Given $6 - 2x + 10 = 20 - 3x - 2$ to find x.
7. Given $5x + 16 = 3x + 100$ to find x.
8. Given $3x + 4 - 80 + x = 22 - 10x$ to find x.
9. Given $3x - 1 + 9 - 5x = 0$ to find x.

Transposing,	$3x-5x=1-9.$
Uniting terms,	$-2x=-8.$
Dividing by -2,	$x=4.$

10. Given $ax - bx = cm$ to find x.

Uniting terms,	$(a-b)x = cm.$
Dividing by $a-b$,	$x = \dfrac{cm}{a-b}.$

11. Given $3ax - am = bn - 3bx$ to find x.

Transposing,	$3ax + 3bx = bn + am.$
Uniting terms,	$3(a+b)x = bn + am.$
Dividing by $3(a+b)$,	$x = \dfrac{bn+am}{3(a+b)}.$

12. Given $mx + ab = nx + 4ab$ to find x.

13. Given $4bx - 5am = 2bx - 10an$.

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14. Given $\frac{1}{4}x + 12 = 22 - \frac{1}{6}x$.

Multiplying by 12,	$3x + 144 = 264 - 2x.$
Transposing,	$3x + 2x = 120.$
Uniting terms,	$5x = 120.$
Dividing by 5,	$x = 24.$

15. Given $\dfrac{x}{a} + \dfrac{x}{b} = c$ to find x.

Clearing of fractions by multiplying each member by ab,*

$$bx + ax = abc.$$

Uniting terms,	$(a+b)x = abc.$
Dividing by $a+b$,	$x = \dfrac{abc}{a+b}$

16. Given $\frac{3}{4}x + 5 = \frac{5}{6}x + 2$ to find x.

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* The student should observe that to multiply a fraction by a number equal to its denominator, he has simply to drop the denominator. Thus to multiply $\dfrac{x}{a}$ by ab, we multiply by the factor a by dropping the denominator, and then multiply the x by the other factor, b.

17. Given $\frac{x+1}{5} + 2 = \frac{2x-3}{3} - 1$ to find x.

18. Given $\frac{7}{8}x - 5 = \frac{9}{10}x - 8$ to find x.

19. Given $\frac{1}{2}x + 10 = \frac{1}{3}x + \frac{1}{4}x + \frac{1}{5}x - 7$ to find x.

20. Given $56 - \frac{3}{4}x = 48 - \frac{5}{8}x$ to find x.

21. Given $25(11x - 8) = 18(12x + 2)$ to find x.

$x = 4$.

22. Given $x(x + 5) = 100 - 20x + x^2$ to find x.

$x = 4$.

23. Given $x - rx = 100$ to find x.

24. Given $42 - \frac{3x+2}{5} = 8 + 5x$ to find x.

Clearing of fractions, $210 - 3x - 2 = 40 + 25x$.

The quantity $\frac{3x+2}{5}$ is the same as one in a parenthesis, and when its denominator is dropped the signs of the terms must be changed, (*109*). The real sign of $3x$, as well as of 2, as they stand in the example, is $+$, the $-$ sign belonging to the whole quantity, and not to either of its terms in particular.

191. Whenever *in clearing an equation of fractions, the denominator of a fraction having a polynomial numerator, and preceded by the — sign, is dropped, the signs of all the terms of the numerator must be changed.*

25. Given $x + \frac{3x-5}{2} = 12 - \frac{2x-4}{3}$ to find x.

$x = 5$.

26. Given $\frac{x+1}{3} - \frac{x+4}{5} = 16 - \frac{x+3}{4}$ to find x.

$x = 41$.

27. Given $3x - \frac{11x-37}{2} = 5 - \frac{2x+6}{5}$ to find x.

$x = 7$.

28. Given $\frac{x-1}{7} = 7 - \frac{23-x}{5} - \frac{4+x}{4}$ to find x.

29. Given $\frac{3x+4}{5} - \frac{7x-3}{2} = \frac{x-16}{4}$ to find x.

30. Given $ax - \frac{a^2-3bx}{a} - ab^2 = bx + \frac{6bx-5a^2}{2a} - \frac{bx+4a}{4}$ to find x.

$$x = \frac{2a(2b^2-5)}{4a-3b}.$$

31. Given $\frac{x}{a} - \frac{dx}{c} + 3ab = 1$ to find x.

$$x = \frac{ac(1-3ab)}{c-ad}.$$

32. Given $\frac{(a-b)x}{2} + \frac{x}{3} = \frac{ab}{4} + a$ to find x.

$$x = \frac{3a(b+4)}{6(a-b)+4}.$$

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33. Given $x^2 = 4$ to find x.

34. Given $3x^2 = 48$ to find x.

35. Given $6x^2 - 2x^2 = 100$ to find x.

36. Given $10x^2 - 36 = 8x^2 + 14$ to find x.

Transposing,	$10x^2 - 8x^2 = 36 + 14.$
Uniting,	$2x^2 = 50.$
Dividing by 2,	$x^2 = 25.$

Extracting the square root of each member, $x = 5$, and -5. We say that if $x^2 = 25$, $x = +5$, because $+5$ squared makes 25. So also $x = -5$, for -5 squared equals 25.

192. *When the square of the unknown quantity equals a known quantity, the unknown quantity itself has two values, one + and the other —, but numerically the same.*

37. Given $35 - \frac{x^2+50}{5} = x^2 - \frac{x^2-10}{3}$ to find x.

$$x = \pm 5.$$

$x = \pm 5$ means that $x = +5$, and $x = -5$. It is read, "x equals *plus* and *minus* 5."

38. Given $ax^2 - b = c$ to find the values of x.

$$x = \pm \frac{1}{a}\sqrt{a(b+c)}.$$

39. Given $ax^2 - 5c = bx^2 - 3c + d$ to find the values of x.

$$x = \pm \frac{1}{a-b}\sqrt{(a-b)(2c+d)}.$$

Equations with Two Unknown Quantities.

193. 1. Given $\left\{\begin{matrix} 3x - 2y = 11 \\ 2x + 3y = 29 \end{matrix}\right\}$ to find x and y.

$$x = 7,\ y = 5.$$

From the equation $3x-2y = 11$, we can find

$$x = \frac{11+2y}{3}.$$

Now substituting this value of x for x in the equation $2x+3y=29$, it becomes

$$\frac{22+4y}{3} + 3y = 29.$$

Solving this last equation as heretofore, we find $y = 5$.

But we had found that $x = \frac{11+2y}{3}$; and now that we know that y is 5, we can substitute this value of y, and have $x = \frac{11+10}{3}$.

194. When we have found the value of one unknown quantity in one of two equations and substituted this value in the other, this unknown quantity is said to be *eliminated* from the latter; *i. e.*, it has disappeared from it.

There are several methods of elimination, but the one given above is adequate to our present purpose, and suggests the following rule for

ELIMINATION BY SUBSTITUTION.

195. RULE.—1. *Having two simple equations between two unknown quantities, find the value of one of the unknown quantities in one of the equations, in terms of the other unknown quantity, and known terms, and substitute this in the other equation.*

2. *Solve the resulting equation as a simple equation with one unknown quantity. Finally substitute this value for the unknown quantity in the value first found for the other.*

2. Given $\left\{\begin{array}{l} 7x - 4y - 3 = 70 \\ \frac{1}{3}x + \frac{1}{4}y - 3 = 4 \end{array}\right\}$ to find x and y.

$x = 15, y = 8.$

3. Given $\left\{\begin{array}{l} \frac{1}{3}x + \frac{1}{5}y = 10 \\ \frac{1}{2}x + \frac{1}{6}y = 11 \end{array}\right\}$ to find x and y.

$x = 12, y = 30.$

4. Given $\left\{\begin{array}{l} 6x + 7y = 33 \\ 8x - 3y = 7 \end{array}\right\}$ to find x and y.

$x = 2, y = 3.$

5. Given $\left\{\begin{array}{l} 16x - 13y - 1 = 30 \\ 24x + 15y - 19 = 200 \end{array}\right\}$ to find x and y.

$x = 6, y = 5.$

6. Given $\left\{\begin{array}{l} 5x - 4y = 19 \\ 4x + 2y = 36 \end{array}\right\}$ to find x and y.

$x = 7, y = 4.$

7. Given $\left\{\begin{array}{l} ax - by = a^2 \\ bx - ay = b^2 \end{array}\right\}$ to find the values of x and y.

$x = \dfrac{a^2 + ab + b^2}{a + b}, y = \dfrac{ab}{a + b}.$

From $ax - by = a^2$, we find

$$x = \frac{a^2 + by}{a}.$$

Substituting this for x in the other equation, we have

$$\frac{a^2b + b^2y}{a} - ay = b^3;$$

whence $$a^2b + b^2y - a^2y = ab^3.$$

Transposing and uniting, $(b^2 - a^2)y = ab^3 - a^2b$, whence

$$y = \frac{ab^3 - a^2b}{b^2 - a^2} = \frac{ab(b-a)}{(b+a)(b-a)} = \frac{ab}{a+b}.$$

But $x = \frac{a^2 + by}{a}$; and substituting the value of y just found, we have $x = \frac{a^2 + \frac{ab^2}{a+b}}{a} = a + \frac{b^2}{a+b} = \frac{a^2 + ab + b^2}{a+b}.$

8. Given $\left\{\begin{array}{l} ax + by = c \\ \frac{x}{b} - \frac{y}{a} = 1 \end{array}\right\}$ to find the values of x and y.

$$x = \frac{ab + c}{2a},\ y = \frac{c - ab}{2b}.$$

9. Given $\left\{\begin{array}{l} x + ay = b \\ ax - by = c \end{array}\right\}$ to find the values of x and y.

$$x = \frac{ac + b^2}{a^2 + b},\ y = \frac{ab - c}{a^2 + b}.$$

10. Given $\left\{\begin{array}{l} ax + by = c^2 \\ a(a + x) = b(b + y) \end{array}\right\}$ to find the values of x and y.

$$x = \frac{b^2 + c^2 - a^2}{2a},\ y = \frac{a^2 + c^2 - b^2}{2b}.$$

11. Given $\left\{\begin{array}{l} ax - by = c \\ x + ay = 3c \end{array}\right\}$ to find the values of x and y.

$$x = \frac{(3b + a)c}{a^2 + b},\ y = \frac{(3a - 1)c}{a^2 + b}.$$

12. Given $p = br$, and $A = b + br$, to eliminate b from the second. Again, to eliminate r from the second. Again, to eliminate r from the first. Again, eliminate b from the first.

To eliminate b from the first, we find the value of b in the second, that is, $b = \frac{A}{1+r}$, and substituting this in the first, have $p = \frac{Ar}{1+r}$.

13. Given $i = prt$, and $A = p + i$,

1. To find A in terms of p, r, and t.
2. To find r in terms of p, i, and t.
3. To find r in terms of A, p, and t.
4. To find t in terms of A, p, and r.
5. To find t in terms of i, p, and r.
6. To find i in terms of r, t, and A.

The last is effected thus: From $i = prt$, we have $p = \frac{i}{rt}$. Substituting this in $A = p + i$, it becomes $A = \frac{i}{rt} + i = \frac{i(1+rt)}{rt}$. Whence $i = \frac{Art}{1+rt}$.

14. Given $l = a + (n-1)d$, and $s = \left(\frac{a+l}{2}\right)n$.

1. To find s in terms of a, n, and d.
2. To find d in terms of l, a, and n.
3. To find n in terms of l, a, and d.
4. To find s in terms of a, l, and d.
5. To find n in terms of s, l, and a.
6. To find a in terms of s, l, and n.
7. To find s in terms of l, n, and d.

The use of equations in the solution of practical problems will be seen in the subsequent sections and chapters. It may, however, be well to allow the student to turn back and complete the solution of the twelve with which this section begins.

SECTION II.

RATIO.

196. There are three methods of comparing numbers, 1st. On the basis of *Equality*, as in the Equation;* 2d. By indicating the difference between them, as when we say one number is 5 greater than another. This method of comparison has sometimes received the name *Arithmetical Ratio.* (See *Arithmetical Progression.*) And, 3d. By telling, or indicating, the quotient of one number *divided* by the other. This is called *Geometrical Ratio*, or usually *Ratio*. We shall always use the word ratio in the latter sense.

197. *Ratio* is the quotient of one number divided by another of the same kind, the former being called the *Antecedent* and the latter the *Consequent.*

The term *Ratio* is also applied to such forms as $6:2$, $\frac{6}{2}$, etc.; that is, to the indicated operation of division, the sign : being an equivalent for $\div$.†

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* The equation is really sufficiently comprehensive to embrace all the other methods; and, in fact, it is the great mathematical instrument for comparison of quantities.

† This double use of the word ratio has given no little trouble to students. That the word is habitually used by mathematicians in both of these ways no one at all conversant with mathematical writing can doubt. Thus when we ask "What is the ratio *of* 12 to 4?" all answer "3;" but when we speak of "the ratio 12 to 4" we have reference rather to the form of the expression $12:4$, or $\frac{12}{4}$, than to its *value*. So also we speak of the ratio *of* a to b as m, for example, but would not say "the ratio a to b is m." Some would avoid this double sense of the word by calling the quotient the *value* of the ratio, and restricting the simple term ratio to the cases where the mere comparison is suggested, as in the second paragraph above.

In this treatise we shall use the form of a fraction to indicate ratio. It is a great pity that any other form of notation has ever come into use; and many European writers wisely disregard it entirely.

198. *A ratio being merely a fraction, or an unexecuted problem in Division, of which the antecedent is the numerator, or dividend, and the consequent the denominator, or divisor, any changes made upon the terms of a ratio produce the same effect upon its value, as the like changes do upon the value of a fraction, when made upon its corresponding terms.*

What effect is produced upon a ratio by multiplying its antecedent? Its consequent? Both? By dividing its antecedent? Its consequent? Both?

199. The term *Reciprocal* is applied to a ratio as to a common fraction, signifying the quotient of 1 divided by the ratio, or the ratio inverted. So also the term *Compound Ratio* has exactly the same meaning as *Compound Fraction, i. e.*, the product of the ratios.

Thus the reciprocal of the ratio a to b is $\frac{b}{a}$, as the ratio itself, called the *Direct Ratio*, is $\frac{a}{b}$. The compound ratio a to b, c to d, e to f, means simply $\frac{a}{b} \times \frac{c}{d} \times \frac{e}{f}$, or, $\frac{ace}{bdf}$, just as the compound fraction $\frac{3}{4}$ of $\frac{5}{7}$ of $\frac{11}{13}$ means $\frac{3 \cdot 5 \cdot 11}{4 \cdot 7 \cdot 13}$. The ratio of 8 to 4 being 2, the reciprocal is $\frac{1}{2}$, etc.

Ex. 1. What is the ratio of 12 to 4? 8 to 2? 7 to 5? m to n? x to y?

2. What effect is produced on the ratio $\frac{4}{8}$ by multiplying the antecedent by 3? By 2? The consequent by 4? By 7? Both terms by 6? By 11? Give the reason in each case.

3. What is the effect of multiplying both terms of the ratio $\frac{a}{b}$ by m? Of dividing both terms of $\frac{an}{bn}$ by n? Of dividing the consequent of $\frac{an}{bn}$ by n? The antecedent?

4. Mention two numbers whose ratio is 7? 5? 2? 11? 30?

5. What number has to 6 the ratio 3? To 7 the ratio 5? To 11 the ratio 13? To a the ratio r? To l the ratio n?

Has 15 the ratio 3 in reference to 6, *i. e.*, is the ratio $\frac{15}{3} = 6$?

6. To what number does 18 bear the ratio 3? To what number does 5 bear the ratio 1? The ratio 5? To what number does ar bear the ratio r? To what number does a bear the ratio r?

7. What is the ratio of $\frac{2}{3}$ to $\frac{5}{7}$? Of $\frac{4}{5}$ to $\frac{3}{10}$? Of $1\frac{1}{2}$ to $2\frac{2}{3}$? Of $\frac{a}{b}$ to $\frac{x}{y}$? Of 10.5 to 2.25? Of 5 to .05? Of .05 to 5?

How do you find the ratio of one number to another? Do you find the ratio of one fraction to another in the same way as of one integer to another?

8. What is the ratio of 1 *ft.* to 1 *yd.*? To 1 *in.*? To 1 rod? What is the ratio of 2 *gal.* to 3 *qt.*? Of $5 to 4 francs? Of 1 mark to 1 franc? Of 1 *lb.* Troy to 1 *lb. Av.*?

Is there any principle involved in answering these questions which is not found in division?

9. Antecedent 12, ratio 3, what is the consequent? Consequent 5, ratio 2, what is the antecedent? Antecedent x, ratio r, what is the consequent? Consequent m, ratio x, what is the antecedent?

10. What simple ratio is equivalent to the compound ratio of 5 to 6, 3 to 4, 2 to 3, 7 to 8, and 12 to 14?

Ans., $\frac{5}{16}$.

11. What simple ratio is equivalent to the compound ratio of a to b, bx to a^2y, xy to y^3, and a^2m to n?

Ans., $\frac{amx^2}{ny^3}$.

SECTION III.

PROPORTION.

200. *Proportion* is an equality of ratios, the terms of the ratios being expressed.

201. Two ratios, at least, are required for a proportion; hence we have two antecedents and two consequents. Of four terms which constitute a proportion, the 1st and 4th are called ***Extremes,*** and the 2d and 3d ***Means.***

The equality of two ratios constituting a proportion may be indicated by the ordinary sign of equality (=), or by the double colon (: :); but the student should understand that the symbol : : means exactly the same as =. We may use the symbol : : in order to distinguish between the conception and reading, as of an ordinary equation, or a proportion.

202. When four numbers are in proportion, as 12, 4, 48, and 16, we say "12 is to 4 as 48 is to 16," meaning thereby that $\frac{12}{4} = \frac{48}{16}$. So if a, b, c, and d are in proportion, we say "a is to b as c is to d," meaning that a is as many times b, as c is times d; that is, that $\frac{a}{b} = \frac{c}{d}$.

203. The principal arithmetical use of proportion is in the solution of problems in which there are three quantities given, two of which sustain the same ratio to each other, that exists between the third and the quantity sought. The method of solving such problems is often called *The Rule of Three.*

Ex. 1. If a man traveled 357 miles in 14 days, how many miles, at this rate, will he travel in 25 days?

To solve such an example by proportion, we observe that the number of miles traveled are in the same ratio as the times of travel, provided the same rate of travel is maintained, *i. e.*, in twice as great a time the man will travel twice as far; in 3 times as great a time, 3 times as far; in half as great a time, $\frac{1}{2}$ as far, etc. Now the ratio of the given time to the supposed time is $\frac{14}{25}$, and as the same ratio exists between the distance as between the times, if we call x the distance traveled in 25 days, the ratio of the distances is $\frac{357}{x}$. But these ratios are equal; hence we have

$$\frac{14}{25} :: \frac{357}{x}.$$

Solving this as a simple equation $x = 637\frac{1}{2}$.

Solve the following, explaining as above.

2. What will 212 bushels of corn cost, if 45 bushels cost \$28.12½?

3. What will 195 boxes of raisins cost, if 212 boxes cost \$238.50?

4. If 25 men can do a piece of work in 27½ days, how many days will it take 5 men to do it?

Will the number of days required to do the work be in the same ratio as the number of men? Will $\frac{1}{2}$ as many men require $\frac{1}{2}$ as much time, or 2 times as much time? Will $\frac{1}{5}$ as many men require $\frac{1}{5}$ as much time, or 5 times as much? *Observe that the time is in the inverse ratio of the number of men employed.* Hence we have

$$\frac{5}{25} :: \frac{27\frac{1}{2}}{x}.$$

5. How many men will it take to do a piece of work in $12\frac{4}{7}$ days which 81 men can do in $6\frac{2}{3}$ days?

6. How many days will it take 42 men to do a piece of work which 81 men can do in $6\frac{2}{3}$ days?

7. If $\frac{2}{3}$ of a farm is worth $240, how much is $\frac{8}{9}$ of it worth?

The ratio of the parts being the same as that of the values, we have $\frac{\frac{2}{3}}{\frac{8}{9}} :: \frac{240}{x}$, or $\frac{2\times 9}{3\times 8}=\frac{240}{x}$, and $x=\frac{\cancel{3}\times\cancel{8}^{4}\times\cancel{240}^{80}}{\cancel{2}\times\cancel{9}_{\cancel{3}}}=320.$

In solving such examples it is usually best to indicate the operations first, and then cancel as much as possible.

8. If $\frac{5}{6}$ of a barrel of flour is worth $5.25, how much is $\frac{5}{12}$ of it worth?

9. How many men will it take to perform a piece of work in $6\frac{2}{3}$ days, which it will take 42 men $12\frac{4}{7}$ days to perform?

10. How many days will it take 81 men to perform a piece of work which 42 men can do in $12\frac{4}{7}$ days?

11. If 95 men consume 27 barrels of flour in 4 weeks, how many barrels will last them 52 weeks?

12. If $\frac{2}{3}$ of a yard of cloth can be bought for $\frac{4}{5}$ of a dollar, what is that per yard?

13. John walks $3\frac{3}{4}$ miles per hour, and William walks $3\frac{1}{3}$ miles per hour. How many hours will it take John to walk as far as William can walk in 8 hours?

14. If $9\frac{1}{2}$ yards of cloth can be bought for $$44\frac{1}{3}$, how many yards can be bought for $$33\frac{1}{4}$?

15. If 12 meters of cloth cost 24 francs, how many dollars will $3\frac{1}{3}$ yards cost?

The proportion is $\dfrac{\dfrac{12 \times 39.37}{12}}{3 \times 3\frac{1}{3}} :: \dfrac{\dfrac{24}{.193}}{x}$, whence $x = \dfrac{3 \times 3\frac{1}{3} \times 24}{39.37 \times .193}$.

16. If a stere of wood cost 10 marks, how many cords can be bought for \$50?

17. At \$7 per acre, how many ares of land can be bought for 1000 marks?

The special value and importance of the preceding method of considering problems in proportion, is that it compels a full understanding of the nature of the process, which the ordinary method does not.

204. PROP. 1.—*The product of the means of a proportion is equal to the product of the extremes.*

DEM.—If a is to b as c to d, we have $\frac{a}{b} = \frac{c}{d}$, and clearing of fractions, $ad = bc$; which proves the proposition, since a and d are the extremes and b and c are the means.

ANOTHER DEMONSTRATION, which shows more clearly *why* the product of the means equals the product of the extremes, is this: The 1st *mean* is the 1st *extreme divided* by the ratio, and the 2d *mean* is the 2d *extreme multiplied* by the ratio. Hence the product of the means is $\frac{\textit{1st Extreme}}{\textit{Ratio}} \times$ 2d *Extreme* $\times$ *Ratio*. In this the Ratio cancels and leaves the product of the Extremes.

205. When *three* numbers are so related that the 1st is to the 2d as the 2d is to the 3d, they are said to be in proportion, and the 2d is called a ***Geometrical Mean*** between the other two. The third term is called a *Third Proportional to the other two.*

206. PROP. 2.—*A Geometrical Mean between two quantities is the square root of their product.*

DEM.—If x is a geometrical mean between a and b, $\frac{a}{x} = \frac{x}{b}$; whence, clearing of fractions, $ab = x^2$; and extracting the square root of each member, $x = \sqrt{ab}$.

Ex. 1. Find the geometrical mean between 4 and 9. 27 and 3. 5 and 11. $\frac{2}{3}$ and $\frac{1}{2}$.

2. Find a third proportional to 7 and 4. To 8 and 12. To 12 and 8. To 4 and 6. To 6 and 4. To $\frac{2}{3}$ and $\frac{4}{5}$. To $\frac{3}{5}$ and $\frac{2}{3}$.

For the last, $\frac{\frac{3}{5}}{\frac{2}{3}} = \frac{\frac{2}{3}}{x}$, or $x = \frac{2}{3} \times \frac{2}{3} \times \frac{5}{3}$; or, by (**204**), we may write at once $\frac{3}{5}x = (\frac{2}{3})^2$, whence $x = \frac{2 \cdot 2}{3 \cdot 3} \times \frac{5}{3}$.

Compound Proportion.

207. Ex. 1. If it cost \$36 to transport 18 tons of goods 60 miles, how much will it cost to transport 54 tons 120 miles?

What is the ratio of the cost of transporting 18 tons, to the cost of transporting 54 tons *the same distance?* *Answer*, $\frac{18}{54}$, or $\frac{1}{3}$. But, if the first is only transported 60 miles, while the second is transported 120, how will this affect the ratio of the cost of the first to the cost of the second? *Answer*, It will make it but $\frac{60}{120}$, or $\frac{1}{2}$ as great as it otherwise would be. Hence we have the ratio of the cost of transporting 18 tons 60 miles, to the cost of transporting 54 tons 120 miles, $\frac{1}{2}$ of $\frac{1}{3}$, and putting x for the cost in the second instance, we have $\frac{1}{2}$ of $\frac{1}{3} = \frac{36}{x}$.

2. If 6 boxes of soap, each holding 9 pounds, cost \$4.59, how much will 11 boxes, each holding 12 pounds, cost?

What would be the ratio of the given cost to the required cost, *i. e.*, $\frac{4.59}{x}$, if only the number of boxes were considered? Now how will this ratio be affected when we consider that the boxes in the first instance contained only 9 *lb.* each, while in the second they contained 12 *lb.* each? *Answer*, In the former case the ratio would be $\frac{6}{11}$, but in the latter only $\frac{9}{12}$ of $\frac{6}{11}$. Hence we have $\frac{9}{12}$ of $\frac{6}{11} = \frac{4.59}{x}$, or $\frac{9}{22} = \frac{4.59}{x}$.

3. If a marble slab 20 feet long, 5 feet wide, and 4 inches thick, weighs 850 pounds, what is the weight of another slab of similar marble, whose length is 16 feet, width 4 feet, and thickness 2 inches?

	LENGTH.	WIDTH.	THICKNESS.
Supposed, or given case,	20 *ft.*,	5 *ft.*,	4 *in.*
Required case,	16 *ft.*,	4 *ft.*,	2 *in.*

Letting $\frac{850}{x}$ be the ratio of the weight in the given case to that in the required case, this ratio would be $\frac{20}{16}$, if only length were considered; but considering the width, the ratio will be $\frac{5}{4}$ as much, or $\frac{5}{4}$ of $\frac{20}{16}$. Again, considering the thickness, the ratio of the given to the required weight is seen to be $\frac{4}{2}$ times as much, or $\frac{4}{2}$ of $\frac{5}{4}$ of $\frac{20}{16}$.

Hence we have $\frac{4}{2}$ of $\frac{5}{4}$ of $\frac{\overset{5}{\cancel{20}}}{\underset{8}{\cancel{16}}} = \frac{850}{x}$, or $\frac{25}{8} = \frac{850}{x}$.

The essential thing is to be sure that the reason for compounding the ratio is seen. Failure in this respect reduces the solution of problems by Compound Proportion to a mere mechanical process.

4. If 8 persons eat \$40 worth of bread in 6 *mo.*, when flour is \$7 a barrel, how many dollars' worth will 24 persons eat in 8⅔ *mo.*, when flour is \$5 a barrel?

Given case,	8 persons,	6 *mo.*,	\$7.
Required case,	24 persons,	8⅔ *mo.*,	\$5.

The proportion is $\frac{7}{5}$ of $\frac{6}{8\frac{2}{3}}$ of $\frac{8}{24} = \frac{40}{x}$

5. If 100 men, by working 6 *hr.* each day, can in 27 *da.* dig 18 cellars, each 40 *ft.* long, 36 *ft.* wide, and 12 *ft.* deep, how many cellars, each 24 *ft.* long, 27 *ft.* wide, and 18 *ft.* deep, can 240 men dig in 81 *da.* of 8 *hr.* each?

	MEN.			LENGTH.	WIDTH.	DEPTH.
Given case,	100,	6 *hr.*,	27 *da.*,	40 *ft.*,	36 *ft.*,	12 *ft.*
Required case,	240,	8 *hr.*,	81 *da.*,	24 *ft.*,	27 *ft.*,	18 *ft.*

Proportion, $\frac{12}{18}$ of $\frac{27}{36}$ of $\frac{24}{40}$ of $\frac{27}{81}$ of $\frac{6}{8}$ of $\frac{100}{240} = \frac{18}{x}$

What ratio would the number of cellars 12 *ft.* deep, which could be dug in a given time, bear to the number 18 *ft.* deep? Would the number be directly, or reciprocally as the depth?

6. If 24 men, by working 8 *hr.* a day, can in 18 *da.* dig a ditch 95 *rd.* long, 12 *ft.* wide, and 9 *ft.* deep, how many men, in 24 *da.* of 12 *hr.* each, will be required to dig a ditch 380 rods long, 9 *ft.* wide, and 6 *ft.* deep?

7. A contractor engaged to pave 15 miles of road in 12 months, and for that purpose employed 100 men. Seven months have now elapsed, and but 6 miles of the road have been completed; how many more men must be employed to finish the work in the time prescribed?

Ans., 110 men.

8. In the manufacture of woolens, it is found that when flannel is made weighing 8.14 *oz.* per yard in length, and shrinks 13.99 per cent. in finishing, 12.41 *oz.* of raw wool have been consumed for every yard of cloth; then how much wool per yard of cloth is consumed when the flannel weighs 7.43 *oz.* and shrinks 8.67 per cent.

Ans., 10.67 ounces.

Partitive Proportion.

208. Ex. 1. Divide 150 into three parts which shall be to each other as 2, 3, and 5.

Such problems are frequently solved by proportion, by considering that the sum of the numbers representing the relation of the parts is to each particular number as the whole number is to that particular part. Thus in this case we should have $\frac{10}{2}=\frac{150}{x}$ for the first part; $\frac{10}{3}=\frac{150}{x}$ for the second part; and $\frac{10}{5}=\frac{150}{x}$ for the third part. But the subject is introduced here to show that the most simple solution is without any proportion at all. Thus call the parts $2x$, $3x$, and $5x$ (which are in the ratio of the numbers 2, 3, and 5), and we have $2x+3x+5x=150$; whence $x=15$, $2x=30$, $3x=45$, and $5x=75$.

It may be well to solve the following in both ways, as the problem is of considerable importance.

2. Divide 35 into 2 parts which shall be to each other as 3 to 4. As 2 to 5. As 1 to 6.

3. Divide 1 into 3 parts which shall be to each other as $\frac{1}{2}$, $\frac{2}{3}$ and $\frac{1}{6}$.

4. Divide 7 into 4 parts which shall be to each other as 3, 5, 8, and 2.

5. Divide $5000 among 3 persons so that the first shall have twice as much as the second and the second twice as much as the third.

6. Divide $100 into 4 parts which shall be to each other as $800, $700, $1000, and $500.

SECTION IV.

PROGRESSIONS.

209. A Progression is a series of terms which increase or decrease by a common difference, or by a common multiplier.

Arithmetical Progression.

210. An Arithmetical Progression is a progression in which the terms increase or decrease by a *Common Difference.*

Thus 3 ·· 7 ·· 11 ·· 15 ·· 19, etc., is an arithmetical progression with a common difference, 4. 12 ·· 10 ·· 8 ·· 6 ·· 4 ·· 2 ·· 0 is an arithmetical progression with a common difference, −2. This common difference is sometimes called *Arithmetical Ratio;* but the term is undesirable.

211. There are ***Five Things*** to be considered in any progression, viz., the first term, the last term, the common difference or the ratio, the number of terms, and the sum of the series. Any three of these five things being given, the other two may be found.

212. The Two Fundamental Formulæ of Arithmetical Progression are:

1. The formula for the *Last Term,* $l = a + (n - 1)\,d.$

2. The formula for the *Sum of the Series,* $s = \left(\frac{a + l}{2}\right) n,$

in which a represents the first term, l the last term, d the common difference, n the number of terms, and s the sum of the series.

To produce the formula for the last term of the series: Since a represents the first term and d the common difference, the second term is $a+d$; the third, $a+d+d$, or $a+2d$; the fourth, $a+3d$; the fifth, $a+4d$; the 6th, $a+5d$; that is, the series is

$$a \cdot\cdot a+d \cdot\cdot a+2d \cdot\cdot a+3d \cdot\cdot a+4d \cdot\cdot a+5d, \text{ etc.}$$

From this we see that any term consists of the first term + the common difference taken as many times as there are terms − 1. Hence for the n^{th} term we have $a+(n-1)d$; or, letting l stand for the n^{th}, or last term, $l=a+(n-1)d$.

To produce the formula for the sum of the series: Since $a \cdot\cdot a+d \cdot\cdot a+2d \cdot\cdot a+3d \cdot\cdot a+4d \cdot\cdot a+5d$, etc., to l represents the series, we have $s=a+(a+d)+(a+2d)+(a+3d)+(a+4d)+$ etc., to l.

Now the term before the last is evidently $l-d$; the term before this, $l-2d$; the term before this, $l-3d$, etc. Hence we may write,

$$s=a+(a+d)+(a+2d)+(a+3d)+ \text{ etc., to } (l-3d)+(l-2d)+(l-d)+l.$$

$$\text{or,}\quad s=l+(l-d)+(l-2d)+(l-3d)+ \text{ etc., to } (a+3d)+(a+2d)+(a+d)+a.$$

Adding, $2s=(a+l)+(a+l)+(a+l)+$ to n terms, or $n(a+l)$;

Whence, dividing by 2, we have $s=\left(\frac{a+l}{2}\right)n$.

213. *Let the student write the rule for finding the last term when the first term, number of terms, and common difference are given. Also for finding the sum when the first term, last term, and number of terms are given.*

It is of the utmost importance that the mathematical student become expert, as soon as possible, in thus *seeing in a formula the process the statement of which we call a rule.* This is exactly what Comte, the most profound writer on the philosophy of mathematics, considers the real province of Arithmetic, viz., making numerical computations from formulæ.

Ex. 1. The first term of an arithmetical progression is 5, the common difference 3, and the number of terms 8, What is the last term? What the sum of the series?

2. Same as above, when the first term is 11, common difference 2, number of terms 7.

3. Find the last term and sum of series when 4 is the first term, 5 the common difference, and 13 the number of terms.

4. What is the 50th odd number? What the sum of the first 50 odd numbers?

5. What is the 35th even number? What the sum of the first 35 even numbers?

6. What is the 31st term of the series $7 \cdot\cdot 11 \cdot\cdot 15 \cdot\cdot 19$, etc.? What the sum of the series?

7. What is the last term of the series $36 \cdot\cdot 34 \cdot\cdot 32$, etc., to 8 terms? What the sum?

Observe that d is -2; whence we have $l = a + (n-1)d = 36 + (8-1)(-2) = 36 - 14 = 22$.

8. What are the 13th term and sum of the series $132 \cdot\cdot 125 \cdot\cdot 118$, etc.?

9. What is the last term of the series $8 \cdot\cdot 5 \cdot\cdot 2$, etc., to 10 terms? What the sum?

$l = a + (n-1)d = 8 + (10-1)(-3) = 8 - 27 = -19$; and the series is $8 \cdot\cdot 5 \cdot\cdot 2 \cdot\cdot -1 \cdot\cdot -4 \cdot\cdot -7 \cdot\cdot -10 \cdot\cdot -13 \cdot\cdot -16 \cdot\cdot -19$. $s = -55$.

QUERY.—Why does the formula $s = \left(\frac{a+l}{2}\right) n$ never give a fractional sum when a, d, and n are integral?

10. Find the 12th term and sum of the series $5 \cdot\cdot 5\frac{1}{3} \cdot\cdot 5\frac{2}{3}$, etc. Of $5 \cdot\cdot 4\frac{2}{3} \cdot\cdot 4\frac{1}{3}$, etc.

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214. PROB. 1.—*Given the extremes* (the first and last terms) *and the number of terms, to find the common difference.*

SOLUTION.—From $l = a+(n-1)d$, find the value of d. Thus, transposing, $(n-1)d = l-a$;* dividing each member by $n-1$, $d = \frac{l-a}{n-1}$.

215. *Let the student write the rule from the formula, in this and each of the following problems.*

216. PROB. 2.—*Given the extremes and the common difference to find the number of terms.*

SOLUTION.—Solving $l = a+(n-1)d$ for n, we have $l = a+nd-d$, $nd = l-a+d$, $n = \frac{l-a}{d} + 1$.

217. PROB. 3.—*Given the last term, number of terms, and common difference, to find the first term.*

As before from $l = a+(n-1)d$, we have $a = l-(n-1)d$.

Ex. 1. What is the common difference of the series of which the first term is 7 and the thirteenth term 43?

2. How many terms are there in the series of which the first term is 8, the last term 85, and the common difference 7?

3. What is the common difference of the series of which 596 is the first term and 491 the twenty-second?

4. How many terms are there in the series of which 12 is the first term, 4 the last, and $-\frac{1}{5}$ the common difference?

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218. PROB. 4.—*Given s, l, and a, to find n.*†

SOLUTION.—Since $s = \left(\frac{a+l}{2}\right)n$, $n = \frac{2s}{a+l}$.

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* Simple transposition gives $-(n-1)d = -l+a$. We may now multiply each member by -1, and obtain $(n-1)d = l-a$. Or we may transpose thus $l-a = (n-1)d$; *i. e.*, changing members, $(n-1)d = l-a$.

† Let these abbreviated statements and solutions be fully expanded in common language, as above.

219. PROB. 5.—*Given s, l, and n, to find a.*

SOLUTION.—Since $s = \left(\frac{a+l}{2}\right)n$, $2s = an + ln$, and $a = \frac{2s - ln}{n}$, or $a = \frac{2s}{n} - l$.

220. PROB. 6.—*Given s, a, and n, to find l.*

Result, $l = \frac{2s}{n} - a$.

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221. PROB. 7.—*Given a, n, and d, to find s.*

SOLUTION.—As neither of the fundamental formulæ, $l = a + (n-1)d$, $s = \left(\frac{a+l}{2}\right)n$, contains all these four quantities, a, n, d, and s, we must combine the two. This we can do by substituting in $s = \left(\frac{a+l}{2}\right)n$, the value of l in terms of a, n, and d, *i. e.*, its value in the first formula. This gives $s = \left(\frac{a + a + (n-1)d}{2}\right)n$, or $s = an + \frac{n(n-1)d}{2}$, as the formulæ required.

222. PROB. 8.—*Given l, n, and d, to find s.*

This is obtained from the two fundamental formulæ in the same manner as the last, and is $s = ln - \frac{n(n-1)d}{2}$, or directly from the preceding formula.

There are 20 such problems, but they cannot all be solved by the simple equation. (See UNIVERSITY ALGEBRA, p. 116.) What is desired here is that *the student fix the two fundamental formulæ in mind, and know how to produce and use them.* It is not desirable for him to commit to memory the formulæ in these problems, nor the rules growing out of them; but he should know how to use the fundamental formulæ in the solution of examples.

Ex. 1. What is the common difference in a series of which the sum is 648, the first term 3, and the last term 78? What the number of terms?

From one or both the fundamental formulæ make the one required by this example, and then substitute in it. The first formula required will contain the value of d in terms of s, a, and l. This formula is $d = \frac{l^2-a^2}{2s-l-a}$.

2. What is the first term and common difference of a series of which the last term is 164, the number of terms 12, and the sum 2100?

3. Form the series of which the sum is 153, the first term 1, and the last term 17?

4. Insert 12 arithmetical means between 20 and 59.

As there are 12 means, the number of terms must be 14.

5. Insert 13 arithmetical means between 3 and $4\frac{1}{2}$.

6. If I travel 100 miles to-day, and 5 miles less on each succeeding day, how far shall I have gone at the end of the 17th day, and what will be the length of the last day's journey?

7. A man commenced walking for exercise, increasing the distance daily by equal additions. The first day he walked 4 miles, and on the fifteenth day he walked $7\frac{1}{2}$ miles. What was the daily increase?

8. The clocks of Venice strike from 1 to 24. How many strokes does one of these clocks make in the day?

9. A person wishes to discharge a debt of \$1125 in 18 annual payments which shall increase in arithmetical progression. How much must his first payment be in order that the last may be \$120? *Ans.*, \$5.

Geometrical Progression.

223. *A Geometrical Progression* is a progression in which the terms increase or decrease by a constant multiplier. If the multiplier is greater than 1 the series is called an *Increasing Progression;* if less than 1, a *Decreasing Progression.* The sign : is used to indicate a geometrical progression.

Thus 3 : 6 : 12 : 24 : 48, etc., is an increasing *Geometrical Progression* in which the *Rate* is 2. 273 : 81 : 27 : 9, etc., is a *Decreasing Geometrical Progression* in which the *Rate* is $\frac{1}{3}$.

224. The constant multiplier by which any term of a geometrical progression is multiplied to produce the next term is called the ***Rate,*** or *Ratio.**

225. In a Geometrical Progression, as in arithmetical, there are *Five Things* to be considered; viz., the first term, the rate, the number of terms, the last term, and the sum of series.

These five things are usually represented thus: a = first term, r = rate, n = number of terms, l = last term, and s = sum of series. Using this notation we have the following *General Form of a Geometrical Progression:* $a : ar : ar^2 : ar^3 : ar^4 : ar^5$, etc., to l.

226. *The two Fundamental Formulæ* of Geometrical Progression are

1. The formula for the last term, $l = ar^{n-1}$.
2. The formula for the sum of the series, $s = \dfrac{lr - a}{r - 1}$.

* It is unfortunate that "ratio" should be used in this sense. It were better to use two terms as the French do, *rapport* = ratio, in proportion; and *raison* = rate, in a progression.

To produce the Formula for the last term, observe that as the second term is the first term multiplied by the rate, the third term the first multiplied by the second power of the rate, the fourth the first multiplied by the third power of the rate, so any term is the first multiplied by the rate raised to a power denoted by the number of terms less 1. Hence if n is the number of terms, we have $l = ar^{n-1}$, as the formula sought.

To produce the Formula for the sum, we have

$$s = a + ar + ar^2 + ar^3 + \text{ etc., to } ar^{n-3} + ar^{n-2} + ar^{n-1}.$$

Multiplying by r,

$$rs = ar + ar^2 + ar^3 + \text{ etc., to } ar^{n-3} + ar^{n-2} + ar^{n-1} + ar^n.$$

Subtracting, $(r-1)\,s = ar^n - a.$

Whence $s = \dfrac{ar^n - a}{r-1}$, or $s = \dfrac{lr-a}{r-1}$, since $ar^n = lr$.

227. *Let the student write the rules deducible from these formulæ.**

Ex. 1. If the first term of a geometrical progression is 5, the rate 2, and the number of terms 8, what is the last term? What the sum of the series?

2. First term 6, rate $\frac{1}{2}$, number of terms 5; find the last term, and the sum of the series.

$l = ar^{n-1} = 6(\frac{1}{2})^4 = \frac{3}{8}$. Again, $s = \dfrac{lr-a}{r-1} = \dfrac{a-lr}{1-r} = \dfrac{6-\frac{3}{16}}{1-\frac{1}{2}} = \frac{93}{8} = 11\frac{5}{8}$.

For a decreasing series it is well to write the formula for s, $s = \dfrac{a-lr}{1-r}$. This is obtained by multiplying both terms of the fraction $\dfrac{lr-a}{r-1}$ by -1.

3. What is the sum of the series of which 2 is the first term, 1458 the last, and 3 the rate.

* This is not so much that the student needs the rule, as that he needs the ability to convert the formula into rules. He should be able to see the arithmetical process in the formula.

4. What is the sixth term of the series $7\frac{1}{4} : 3\frac{5}{8} : 1\frac{13}{16}$, etc.? What is the sum?

5. What is the first term, if 1280 is the last, 2 the rate, and 9 the number of terms? *Ans.*, 5.

6. What is the sum of the following series: $1 + (1 + r't) + (1 + r't)^2 + (1 + r't)^3 +$ etc., to $(1 + r't)^{n-1}$?

Here $a = 1$, $r = (1 + r't)$, and $l = (1 + r't)^{n-1}$. Hence we have $s = \frac{lr - a}{r - 1} = \frac{(1 + r't)^{n-1}(1 + r't) - 1}{(1 + r't) - 1} = \frac{(1 + r't)^n - 1}{r't}$. Observe that to multiply $(1 + r't)^{n-1}$ by $1 + r't$, we have but to consider that in $(1 + r't)^{n-1}$ there are $n-1$ factors, each $1 + r't$; whence multiplying by $1 + r't$ simply introduces another such factor and makes the product n factors, each $1 + r't$, or $(1 + r't)^n$.

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228. PROP.—*The formula for the sum of an **Infinite Decreasing Geometrical Progression** is* $s = \frac{a}{1 - r}$.

DEM.—Since the terms are growing less and less, when the series is extended to infinity l becomes 0; whence the formula $s = \frac{a - lr}{1 - r}$, becomes $s = \frac{a}{1 - r}$.

Ex. 1. What is the sum of $1 : \frac{1}{2} : \frac{1}{4}$, etc., to infinity, *i.e.*, extended forever? *Ans.*, 2.

2. What is $\frac{1}{2} + \frac{3}{8} + \frac{9}{32}$, etc., to infinity?

3. What is $14 + 2 + \frac{2}{7}$, etc., to infinity?

4. What is the value of $.\dot{3}\dot{2}$?

The repetend $.\dot{3}\dot{2}$ is $\frac{32}{100} + \frac{32}{10000} + \frac{32}{1000000}$, etc., *i. e.*, an infinite decreasing geometrical progression, in which the first term is $\frac{32}{100}$, and the rate $\frac{1}{100}$. Hence $s = \frac{\frac{32}{100}}{1 - \frac{1}{100}} = \frac{32}{99}$. (See p. 28.)

5. What is the value of $.\dot{2}$? Of $.\dot{0}\dot{3}$? Of $.0\dot{3}$? Of $.0\dot{2}\dot{5}$? Of $.\dot{0}2\dot{5}$?

229. From the two fundamental formulæ, $l = ar^{n-1}$, and $s = \dfrac{lr - a}{r - 1}$, eighteen others may be deduced, which enable us to find any two of the five terms, a, l, r, n, s, when the other three are given; but to deduce several of these requires a knowledge of equations higher than the simple equation, and some require a knowledge of logarithms. (See UNIVERSITY ALGEBRA, p. 119.) The following may afford the student useful exercise:

1. Given $a,\ r,\ s,$ $\quad l = \dfrac{a + (r-1)s}{r}.$

2. Given $r,\ n,\ s,$ $\quad l = \dfrac{(r-1)sr^{n-1}}{r^n - 1}.$

3. Given $a,\ r,\ n,$ $\quad s = \dfrac{a(r^n - 1)}{r - 1}.$

4. Given $r,\ n,\ l,$ $\quad s = \dfrac{lr^n - l}{r^n - r^{n-1}}.$

5. Given $r,\ n,\ l,$ $\quad a = \dfrac{l}{r^{n-1}}.$

6. Given $r,\ n,\ s,$ $\quad a = \dfrac{(r-1)s}{r^n - 1}.$

7. Given $r,\ l,\ s,$ $\quad a = rl - (r-1)s.$

8. Given $a,\ l,\ s,$ $\quad r = \dfrac{s - a}{s - l}.$

The 1st is deduced from $s = \dfrac{lr-a}{r-1}$, since this contains the four quantities under consideration, viz., a, r, s, and l.

The 2d requires the combination of the two formulæ, since neither one of them contains the four quantities, r, n, s, and l. To obtain this, eliminate a from the formulæ, $s = \dfrac{lr-a}{r-1}$, and $l = ar^{n-1}$.

From the first of these, $a = lr-(r-1)s$. Substituting this in the second, we have $l=[lr-(r-1)s]r^{n-1}=lr^n-(r-1)sr^{n-1}$, whence $lr^n-l=(r-1)sr^{n-1}$, and $l=\frac{(r-1)sr^{n-1}}{r^n-1}$.

It is not desirable that the student attempt to fix in memory any of these formulæ, except the two fundamental ones; but he should know how to deduce others from these, as occasion requires.

Ex. 1. Given $a=7$, $r=3$, and $l=1240029$. What is s?

2. Given $r=3$, $n=7$, $s=1093$, to find l.

3. Given $a=1$, $l=2048$, $s=4095$, to find r.

4. A gentleman married his daughter on New Year's day, and gave her husband 1 dollar toward her portion, and was to double it on the first day of every month during the year. What was her portion? *Ans.*, $4095.

5. A king in India, named Sheran, wished (according to the Arabic author Asephad) that Sessa, the inventor of chess, should himself choose a reward. He requested the king to give him 1 grain of wheat for the first square, 2 grains for the second square, 4 grains for the third square, and so on; reckoning for each of the 64 squares of the board twice as many grains as for the preceding. Sheran was angry at a demand apparently so insignificant; but when it was calculated, to his astonishment it was found to be an enormous quantity. What was the number of grains of wheat, and what was its worth at $1.50 per bushel, reckoning 7680 grains to a pint?

Ans., 18 446 744 073 709 551 615 grains.
37 529 996 894 754 bushels.
$56 294 995 342 131.

To solve this, 2 has to be involved to a very high power. For an expeditious method of doing this, see (*89*).

SYNOPSIS.*

COMPARISON OF NUMBERS.

- **Methods of comparing Numbers (*196*).**
- **Equation.**
  - What. Members.
  - Known and Unknown Quantities.
  - Transformations.
    - *Number of.—What.*
    - *Method.*
    - *Purpose.*
    - *Axioms.*
  - Elimination.
- **Ratio.**
  - What.
  - Terms of.
  - Same as fraction.
  - How affected by changes of terms.
  - Reciprocal.
  - Compound.
- **Proportion.**
  - What.
  - Number of Ratios.
  - Extremes. Means.
  - Rule of Three. The essential relation between the three numbers.
  - Product of extremes = Product of means.
  - Geometrical Mean. / Third Proportional. — *To find.*
  - Compound Proportion.
- **Progression.**
  - What.
  - Things to be considered.
  - Kinds.
  - Arithmetical.
    - *General form.*
    - *Fund. Formulæ. Produced.*
  - Geometrical.
    - *General form.*
    - *Fund. Formulæ. Produced.*

* For a more complete Synopsis, see COMPLETE SCHOOL ALGEBRA.

CHAPTER VII.

BUSINESS ARITHMETIC.

SECTION I.

PERCENTAGE.

230. So large a proportion of business transactions are based upon percentage, that this subject is very properly considered as the foundation of our *Commercial*, or *Business Arithmetic.*

[The author would urge the importance of having *double solutions* given to all examples: 1st. By the formulæ; and 2d. By the elementary analysis. The former gives breadth of view and the familiarity with formulæ which is necessary to ability to investigate, and the latter gives clearness of conception of the nature of the transaction, and is essential in all elementary presentation of the subjects. Especially should all who propose to be teachers become expert in both. Greater prominence is given to the formulæ in this treatise, because the student is assumed to be somewhat familiar with the common analysis.]

231. *Per Cent* means *By the Hundred.* The character % is used as a substitute for the words *per cent.*

232. *Rate** is the number by which we multiply to

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* It seems scarcely admissible to use the term *Rate per cent* in this sense, but we may so use *Rate*; in fact, this is the common meaning of the word *rate* in mathematics. An allowance of 7 on a hundred is not at a rate of .07 *per cent*, although it is at a *rate* of .07; the *Rate per cent* is 7.

obtain any required per cent of a given number. *Rate per cent,* therefore, means rate by the hundred.

The result obtained by taking a certain per cent of a number is called *the Percentage.* The term *Percentage* is also used as a general designation for all processes involving this method of reckoning by the hundred.

233. The ***Base*** is the number upon which the percentage is estimated.

234. The ***Amount*** is the sum of the base and percentage.

ILLUSTRATIONS.—An allowance, or estimate of 7 by the hundred, or 7 for every hundred on any number, is an allowance or estimate of 7%. The *rate per cent* is 7. Since taking 7 of every hundred is taking .07, when the *rate per cent.* is 7 the *rate* is .07, etc.

Problems in Percentage.

235. There are *Four Things* considered in Percentage, viz., the *Base,* the *Rate,* the *Percentage,* and the *Amount.*

236. The ***Fundamental Formulæ*** of Percentage are: 1. $p = br$, and 2. $A = b + br$, or $b(1 + r)$, in which b represents the *base,* r the *rate,* p the *percentage,* and A the *amount.*

To produce $p=br$, we have but to remember the definitions of *base, rate* and *percentage* (***233, 232***). This is illustrated by the problem, To take 8% of 345. Now 8% of anything means .08 of it; hence 8% of 345 is 345 × .08, and the *percentage* = 345 × .08. In the formula $p = br$, r is hundredths.*

* In the author's COMPLETE SCHOOL ALGEBRA, in these formulæ r is used to represent the *Per Cent,* or *Rate per cent;* here it represents the *rate.*

To produce the second formulæ we have but to remember the definition of *amount* (*234*). From this, $A = b + br$, since b is the base and br the percentage, and the sum of these two is the amount.

237. There being *four things, p, b, r,* and *A* involved in these two formulæ, any two of which being given, the other two may be found directly therefrom, there are *twelve* possible cases, including the fundamental ones.

238. Let the student deduce 3 to 11 from the fundamental formulæ, 1 and 2.

1. $\boldsymbol{p = br.}$

2. $\boldsymbol{A = b + br,}$ or $\boldsymbol{b(1 + r).}$

3. $p = A - b.$	6. $b = \frac{p}{r}.$	10. $r = \frac{p}{b}.$
4. $p = \frac{Ar}{1 + r},$ or $\frac{A}{1 + r}r.$	7. $b = \frac{A}{1 + r}.$	11. $A = b + p.$
	8. $r = \frac{A}{b} - 1.$	12. $A = \frac{p(1 + r)}{r},$ or $\frac{p}{r} + p.$
5. $b = A - p.$	9. $r = \frac{p}{A - p}.$	

Do not deduce these 10 one from the other, but deduce each directly from (1) or (2), or from both. Thus to deduce (9), find the value of b in (1), *i.e.*, $b = \frac{p}{r}$, and substitute this value in (2), obtaining $A = \frac{p}{r} + p$; whence $r = \frac{p}{A - p}.$

239. *State each of the above as a rule in the ordinary way.*

Thus the 7th gives. *To find the base when the amount and rate are given, divide the amount by 1 + the rate.*

240. *As a second class of exercises on these formulæ, justify each one by the ordinary method,* that is, by the common arithmetical analysis.*

Thus, to justify $b = \frac{A}{1+r}$, we observe that a base 1 at $r\%$ gives an amount 1 plus $r\%$ of 1, which is $1+r$. Hence amount A corresponds to as many units of base as $1+r$ is contained times in A, *i. e.*, $\frac{A}{1+r}$.

To justify $A = \frac{p(1+r)}{r}$, we may observe $\frac{p(1+r)}{r}$ is $\frac{p}{r}+p$. But $\frac{p}{r}$ is the base, since 1 of base gives r of percentage, p percentage gives as many units of base as r is contained times in p. Now since $\frac{p}{r}$ is base, $\frac{p}{r} + p$ is base + percentage, which by definition is amount.

In a similar manner proceed with the others.

241. Solve each of the following examples both by the appropriate formula, and by an elementary analysis. In solving by the formulæ, *do not try to remember anything but the two fundamental ones.* In each particular case determine first what kind of formula is needed, and then produce it from the two fundamental ones.

1. Given amount 270, and rate per cent 8, to find the percentage and the base.

1. *To solve by the Formulæ.*—We observe that we want a formula which gives the percentage in terms of the amount, and rate. Now neither (1) $p = br$, nor (2) $A = b+br$ does this. We must therefore combine them and eliminate the quantity b, which we do not want. From $p = br$, we have $b = \frac{p}{r}$; and substituting this in $A = b+br$, it becomes $A = \frac{p}{r}+p$, from which we have $p = \frac{Ar}{1+r}$, as the formula needed. Hence $p = \frac{Ar}{1+r} = \frac{270 \times .08}{1.08} = 20$.

* The student of this book is presumed to be familiar with percentage as presented in ELEMENTARY ARITHMETIC; but this class of exercises will afford an excellent review.

2. *To Analyze Elementarily.*—Every 1 of base is 1.08 of amount. Hence amount 270 corresponds to $\frac{270}{1.08}$ of base, and as the percentage is .08 of the base, we have the percentage $\frac{270}{1.08} \times .08 = 20$. In this analysis we have found the base $\frac{270}{1.08}$ which is 250.

2. Given base 250, and amount 270, to find percentage and rate.

In order to obtain the benefit designed to be derived from the solution of these examples:

1. Attempt to remember only the *two fundamental formulæ.*
2. In any given case produce the particular formula needed, and always obtain the thing sought *directly* from the *two given things.*
3. Analyze each example, as well as solve it by means of the formulæ.

3. Given rate per cent 8, and percentage 20, to find base and amount.

4. Given amount 126.44, and percentage 10.44, to find base and rate.

5. Given amount 126.44, and rate .09, to find base and percentage.

6. Given percentage 202.8, and rate per cent 13, to find base and amount.

7. Given rate .13, and amount 1762.8, to find percentage directly. To find base.

8. Given amount 157.17, and rate per cent $\frac{3}{4}$, to find percentage directly. To find base.

9. Given base 3580, and rate per cent $1\frac{1}{2}$, to find amount directly without first finding percentage.

10. Given rate $1.12\frac{1}{2}$, and base 1342, to find amount directly, *i. e.*, without first finding the percentage.

11. Given amount 2851.75, and rate per cent $112\frac{1}{2}$, to find base directly.

12. Given rate 3, percentage 600, to find amount directly.

242. It frequently happens that the percentage is to be *subtracted* from the base instead of added to it, thus giving a *Difference* instead of an *Amount.* In order to treat such cases, it is only necessary to remember that formula (2) becomes $D = b - br$, using D for *difference.*

Ex. 1. Given $D = 218.40$, and 9%, to find the base and the percentage.

Since (2) contains D, b, and r, we deduce from it

$$b = \frac{D}{1-r} = \frac{218.40}{1-.09} = \frac{218.40}{.91} = 240.$$

Again, neither (1) $p = br$, nor (2) $D = b - br$, contains the three quantities D, r, and p; we combine them and obtain

$$p = \frac{Dr}{1-r} = \frac{218.4 \times .09}{.91} = 21.60.$$

To analyze for the base, consider a deduction of 9% from 1 of base leaves a difference of $1 - .09 = .91$. Hence a difference of 218.40 requires as many units of base as .91 is contained times in 218.40, *i. e.*, $\frac{218.40}{.91} = 240$.

To analyze for percentage, having obtained the base, 240, we consider that the percentage is .09 of the base.

2. Given difference 328.70, and percentage 17.30, to find base and rate.

3. Given base 346, and percentage 17.30, to find rate and difference.

4. Given rate .05, and percentage 17.30, to find difference and base.

In this example, is it necessary to find the base first? Can any one of the four quantities involved in problems in percentage be found directly from any two given ones?

5. Given percentage .5665, and difference 14.8835, to find rate and base.

6. Given rate per cent $3\frac{2}{3}$, and difference 14.8835, to find base and percentage.

7. Given base 15.45, and difference 14.8835, to find rate and percentage.

8. Given base $\frac{2}{3}$, and difference $\frac{8}{15}$, to find rate and percentage.

9. Given percentage $\frac{2}{15}$, and rate per cent 20, to find base and difference.

10. Given rate per cent 150, and difference $\frac{5}{12}$, to find percentage and base.

11. Obtain each of the following from the formulæ $p = br$, and $D = b - br$:

(3.) $p = b - D$;	(8.) $r = 1 - \frac{D}{r}$;
(4.) $p = \frac{Dr}{1-r}$;	(9.) $r = \frac{p}{D+p}$;
(5.) $b = D + p$;	(11.) $D = b - p$;
(7.) $b = \frac{D}{1-r}$;	(12.) $D = \frac{p(1-r)}{r}$, or $\frac{p}{r} - p$.

12. Deduce all the formulæ for percentage when the *difference* instead of the amount is involved, by changing the signs of the formulæ involving the amount.

This is a very important exercise to familiarize the student with the nature of such transformations. In this case the *rate* becomes a rate of *decrease*, and hence is $-$. So also since percentage is base multiplied by rate, and the latter factor is $-$, p is $-$. Applying these principles to the formulæ in (*238*), and putting D for A, the changes are effected.

Thus $p = br$ will remain the same, since $-p = -br$ is the same as $p = br$. $A = b + br$ becomes $D = b - br$, by attributing the $-$

sign to the factor r. (3), which is $p=A-b$, becomes $-p=D-b$, and multiplying each member by -1 to change the sign of p, $p=b-D$. (12), which is $A=\frac{p(1+r)}{r}$, becomes $D=\frac{-p(1-r)}{-r}$, or $D=\frac{p(1-r)}{r}$, since changing the sign of numerator and denominator is equivalent to multiplying or dividing both by -1. In like manner deduce all the others. It will afford a check on the correctness of the work to deduce the particular formula by thus changing signs, and also from the two formulæ, $p=br$, and $D=b-br$. Of course the results must agree.

13. What per cent of a number is $\frac{1}{2}$ of it? $\frac{1}{3}$ of it? $\frac{2}{5}$? $\frac{3}{4}$? $1\frac{1}{2}$ times the number? 2 times?

14. 20% is what part of a number? 50%? $33\frac{1}{3}$%? 100%? 200%? 75%? 10%?

15. 32 is what per cent of 400? $\frac{2}{3}$ of $\frac{5}{6}$? 38.88 of 648? $\frac{3}{4}$ of 11? $1\frac{1}{2}$ of 12?

What formula covers all such cases as these in Ex. 15? From this formula, what is the rule for finding what per cent. one number is of another?

SECTION II.

BUSINESS CALCULATIONS BASED ON SIMPLE PERCENTAGE, WITHOUT THE ELEMENT OF TIME.

243. Business Calculations based on percentage are of two classes: 1. Those which do not involve the element of *Time* in the computations; and 2. Those which involve the element of *Time* in the computations.

Of the 1st class are the simpler problems of *Profit* and *Loss, Commission, Brokerage, Stocks, Insurance, Taxes,* and *Duties.* Of the 2d class are *Interest, Discount, Annuities,* including many problems in *Insurance, Exchange,* and *Equation of Payments and Accounts.*

244. Problems of the 1st class are solved directly by the principles, or formulæ of the preceding section.

[We shall defer the treatment of Insurance till the next section, since many of its most important problems involve the element of *Time;* and Stocks will for a similar reason be deferred till after Discount and Exchange.]

Profit and Loss.

245. Profit, or ***Gain,*** is the excess of what is received for an article over its total cost. ***Loss*** is the excess of the total cost of an article over what is received for it.

[Observe the suggestions in ***241.***]

Ex. 1. At what must cloth which cost \$3.50 per yard be sold to gain 20%?

2. At what must sugar that cost 10*c.* per pound be sold so as to yield a profit of 15%? What to yield 25%? What to yield 10%?

3. Bought a horse for \$185 and sold it for \$222. What per cent did I make on the investment?

To what in percentage does the \$185 in this problem correspond? To what the \$222? Does the formula $p = br$, or $A = b + br$ solve the example? One or the other of these formulæ, or both combined, will solve all questions in simple percentage. In this example, \$222 is the amount, and \$185 is the base; hence $A = b + br$ becomes $r = \frac{A-b}{b} = \frac{222-185}{185} = .20$. Hence the gain was 20%.

4. A man sold a house at a profit of $33\frac{1}{3}\%$, and thereby gained \$7500; required the cost and selling price?

Here we have $p = \$7500$, and $r = .33\frac{1}{3}$, and are required to find b, the cost, and A, the selling price. $p = br$ gives the former, as $b = \frac{p}{r} = \frac{7500}{.33\frac{1}{3}} = \frac{7500}{\frac{1}{3}} = 3 \times 7500 = 22500.$

Now $A = b + p = 22500 + 7500 = 30000.$

5. Sold 5000 acres of land at $\$3\frac{1}{4}$ an acre, and thereby gained 22%; what was the cost?

Here $A = 5000 \times 3\frac{1}{4}$. Having given A and r, we wish to find b. Hence we use $A = b + br$, or $b = \frac{A}{1+r} = \frac{5000 \times 3\frac{1}{4}}{1.22}$.

6. A speculator lost \$1950 on a lot of flour, which was 20% of the cost; required the cost.

Formula, $p = br$, whence $b = \frac{p}{r} = \frac{1950}{.20} = \frac{1950}{\frac{1}{5}} = 5 \times 1950.$

Or, as $.20 = \frac{1}{5}$, \$1950 is $\frac{1}{5}$ the cost.

7. Having lost 12% of my property, which was \$420, how much had I remaining?

The remainder was the *difference*, and what I lost was the *percentage;* hence we have p and r given, to find D. Neither formula has these three quantities in it. But from (1) we have $b = \frac{p}{r}$, and substituting in (2) we obtain

$$D = \frac{p(1-r)}{r} = \frac{420(1-.12)}{.12} = \frac{420 \times .88}{.12} = 35 \times 88.$$

In applying these formulæ, cancel as much as possible. In this case, dropping the decimal point, which multiplies numerator and denominator by 100, and canceling 12, we have 35×88.

To analyze this by the elementary method, we should first find the base, or capital invested. By the formulæ we have no need of finding the base.

8. Bought 4 barrels of sugar, each containing 195 pounds, at \$11.70 a barrel, and sold it for $8\frac{3}{5}$ cents a pound. How much was the gain?

Of course gain or loss is not necessarily estimated *by the hundred*, that is, at a *per cent.*

9. Bought cloth for \$1.50 a yard; how much will be the loss per cent if I sell it at \$1.25 a yard?

10. A. sold pork for $87\frac{1}{2}\%$ of its cost, and thereby lost $\$3.33\frac{1}{3}$ on a barrel; required the cost per barrel.

11. A merchant lost 15% on his old stock of goods; how much did he lose on those that cost $12\frac{1}{2}$ cents, $\$6\frac{2}{3}$, $38\frac{1}{3}$ cents, $\$33\frac{1}{3}$, and $\$18\frac{4}{5}$?

12. A speculator bought \$15600 worth of flour and sold it to a merchant at a gain of $33\frac{1}{3}\%$. and the merchant sold it at a loss of 25%; how much more did the speculator receive for the flour than the merchant?

13. If I sell $\frac{3}{4}$ of an article for the cost of the whole of it, what % gain do I make on the part sold?

Suppose I paid $\$4n$ for the article. For what I sold I paid $\$3n$, which is therefore the *base*, and received for this $\$4n$, which is the *amount* on the base of $\$3n$. From $A = b + br$,

$$r = \frac{A-b}{b} = \frac{4n-3n}{3n} = \frac{1n}{3n} = \frac{1}{3}.$$

Hence the gain on the part sold was $33\frac{1}{3}\%$.

14. If I sell $\frac{3}{4}$ of an article for what $\frac{1}{2}$ of it cost me, what % do I lose on the part sold?

15. If $\frac{4}{5}$ of the buying price equals the selling price, what is the loss per cent?

16. If $\frac{4}{5}$ of the selling price equals the buying price, what is the gain per cent?

17. In marking goods, what fractional part of the cost must be added to the cost to give a gain of 5%? Of 10%? Of 20%? Of 25%? Of 50%? Of 100%? Of $12\frac{1}{2}$%?

18. At what price per bushel must 3000 *bu.* of wheat be sold to gain $4\frac{6}{11}$%, if 2000 *bu.* of it cost $1.09 per bushel, and 1000 *bu.*, $1.12?

19. If by selling coffee at 27*c.* per pound there is a gain of 20%, what must be the selling price to give a gain of $16\frac{2}{3}$%? A gain of 30%? Of 10%?

20. Having purchased a house for $4500, and spent $500 in improvements, I sold it for $5800. What % did I make on my investment?

Given $b = \$5000$, and $A = \$5800$, to find r.

21. A merchant, by selling 40 meters of cloth for $164, lost 20 per cent; what did it cost per meter?

22. If 3 decaliters and 5 liters of molasses cost $7.00, how must it sell per litre to gain 20%?

23. B lost 5 per cent by selling a hectoliter of turpentine, which cost $15; for what did he sell it a liter?

24. Sold cloth which cost me 8 francs per yard at 8 marks per yard. What % did I make?

Commission.

246. An ***Agent, Broker,*** or ***Commission Merchant,*** is a person who does business for another.

Commission or ***Brokerage*** is the percentage paid an agent, broker, or commission merchant, and is estimated at a certain rate % on the amount of business done.

The distinction between Commission and Brokerage is not very clearly defined; but in a general way it may be said that the term Broker is more exclusively applied to persons dealing in money, stocks, exchanges, or other more characteristically monetary matters, while a Commission Merchant deals in some other kind of property. The term *Agent* seems to be getting into use as a general term covering all classes of business men who render service in business affairs for others, away from the central office.

247. The ***Amount*** of money *received* or *expended* in behalf of another is usually the *base* on which commission is reckoned; except in case of dealings in stocks and exchange, in which cases it is customary to estimate brokerage on the *par value* of the paper.

[In solving the following, observe the directions in **241**.]

Ex. 1. A dealer in real estate sold a farm for Mr. A., charging him 5%. His commission was \$375. For what did he sell the farm?

By the formula. $r = .05$, $p = 375$, to find b.

Analyzed.—On every \$1 received for farm, the agent receives \$.05. Hence if he receives \$375, the value must have been $\frac{375}{.05}$.

2. My agent in New York bought a bill of goods for me amounting to \$3500. What must I remit, allowing him a commission of $2\frac{1}{4}$% ?

Given b and r, to find A.

3. I received \$3700 to invest in land for Mr. B., first deducting my commission of 8%. How much can I expend for land?

Of course I am not to take commission on my own pay. The \$3700 is A, and not b. $b = \frac{A}{1+r}$.

Analyzed.—For every \$1's worth of land I buy for him, Mr. B. must send me \$1.08. Hence every \$1.08 buys \$1's worth of land, or I buy $\frac{3700}{1.08}$ dollar's worth.

4. A receiver of taxes collected \$75450. What shall he pay into the treasury, his commission being 5%?

In this case \$75450 is the base, *b*. Why?

5. I sent a note of \$2500 to a lawyer in Hudson, with instruction to secure what he could upon it, as I understood that the firm against which the note was, had gone into bankruptcy. He secured $62\frac{1}{2}$% on the face of the note, and charged me 5% commission. How much did he remit to me?

The base is $62\frac{1}{2}$% of 2500. Why? (See *247*.)

6. What amount of goods can be bought for \$8758.25, allowing $2\frac{1}{2}$ per cent commission?

7. An agent received \$5650 to invest in wheat, at a commission of $3\frac{1}{3}$ per cent; how much was expended in wheat, and what was the agent's commission?

8. A Michigan merchant sent to a commission merchant in Chicago 12 tons of maple sugar during the season. The Chicago merchant paid railroad charges at 50*c.* per *cwt.*, and \$3.00 in all for cartage. His commission was $2\frac{1}{2}$ per cent. What would he remit the Michigan merchant, he having sold the sugar at 25*c.* per pound?

Ans., \$5727.

[This subject will be extended under subsequent heads, especially in connection with stocks.]

State and Local Taxes.

248. A ***Tax*** is money required by the government to be paid by the people of the country for the support of government, or for public enterprises.

The expenses of the State, County, Township, Village, City, or School District, are in the main paid by taxes upon the property included within the particular bounds. Licenses and fines are also made to contribute to this end. The expenses of the *General Government* (United States Government) are paid by what are called *Duties* (see *253*), or by taxes on certain kinds of business, charges for certain special service (as carrying the mails), etc. The latter constitutes what is called the *Internal Revenue.* The expenses thus paid are officers' salaries, making public improvements (as building common roads and bridges), improving navigation, building public buildings, supporting the army, carrying the mails, etc.

The money raised for the uses of the General Government constitutes the United States *Revenue.* All other governmental expenses are embraced under the designation *State and Local Taxes.* The latter are treated here. For the former, see (*252*).

249. For purposes of taxation, property is considered as *Real* and *Personal.* ***Real Property***, or *Real Estate*, is land and houses. ***Personal Property*** is such as furniture, goods, vessels, notes, mortgages, stocks, etc.

An estimated value has to be put on all taxable property, in order to taxation. The value at which *Real Estate* shall be reckoned for taxation is determined by the officers of government, and is generally not more than $\frac{1}{2}$ or $\frac{1}{3}$ of its real value. The value of personal property may be given in by the owner under oath, or if he does not choose to do this it will be fixed by the officers of government.

250. An ***Assessor*** is an officer whose duty it is to ascertain and make out a list of the taxable property in a given district. This list is called an *Assessment Roll.* A ***Collector*** is an officer who collects the taxes.

251. There are in some of the States what are called *Poll Taxes.* A poll tax is usually a small sum (75c. or $1) required of every male over 21 years of age.

The gross amount of money necessary to be raised in any given year for State purposes is determined by the Legislature; for county purposes by the *Board of Supervisors*, or the *County Commissioners*, or the highest authority in the county under some other name; for city purposes by the *Common Council*, etc. When in any given case this is determined and the assessors have made out a list of all the taxable property, the taxes to be paid by individuals can be computed.

Ex. 1. The citizens in a certain school district decide to build a new school-house on a new site. The estimated cost of the site is $300, and of the house $10,000. It is estimated that they can realize $800 for the old house and lot. The assessment roll shows that the taxable property of the district is $750,000. What is the rate per cent to be levied on the valuation? What the tax of a man whose property is assessed at $10,000? What a man's tax whose property is assessed at $2,200?

2. In the above district, what amount would a tax of $2\frac{1}{2}\%$ on the valuation raise, allowing 5% for collecting?

What would be the total amount raised? What the commission for collecting? What the amount available for the use of the district?

3. Suppose in the above district it was desired to raise $3000 for the current expenses of the schools. Allowing 5% for collecting and 3% as uncollectible, what per cent must be levied for this purpose?

4. The total of the State tax in Michigan as apportioned on September 25, 1874, was $903,434.50, and its items were as follows: Agricultural College, $28,602; Kalamazoo

Insane Asylum, $60,000; New Insane Asylum, $100,000; General Purposes, $300,000; Flint Asylum for Deaf and Dumb and Blind, $46,000; Military Fund, $33,382.50; State Capitol, $200,000; State Prison, $50,000; State Public School, $20,000; State Reform School, $33,950; University Aid, $31,500.

The total valuation of property in the State was $630,000,000. From these facts determine the following:

1st. What was the total tax on $1?

2d. A man who was worth $12,000 would be assessed on about $4,000. How much would such a man pay towards the State Capitol? Towards the University? Towards the Agricultural College? What would be his total State tax?

3d. A man who was worth $4000 would be assessed on about $1350. How much would he pay for the State Capitol? For the military fund? For educational purposes, including Agricultural College, State Public School, and the University? What for each of these separately?

4th. What is the University tax on $1 valuation? What the tax for general purposes?

5. The assessed value of property in the State of Illinois for 1873 was $1,355,401,317. The total tax assessed was $21,963,821.29. Of the latter $5,023,609.50 was for State purposes, $5,533,091.20 for county purposes, $1,583,942.32 for city purposes, and $9,823,178.27 for town, district, and other local purposes. The school tax for this year was $999,587.91.

6. What was the tax on $1 evaluation in the State of Illinois for 1873?

7. From the data in 5 and 6 fill out the following

TABLE.

PROP.	TAX.	PROP.	TAX.	PROP.	TAX.	PROP.	TAX.
$1		$10		$100		$1000	
2		20		200		2000	
3		30		300		3000	
4		40		400		4000	
5		50		500		5000	
6		60		600		6000	
7		70		700		7000	
8		80		800		8000	
9		90		900		9000	

8. From the above table, when filled out, determine by mere addition what a man's tax would be who was taxed on $5680 real estate and $1050 personal property. A man who was taxed on $548 real estate and $85 personal property. A man taxed on $8572 real estate, and $12765 personal property.

9. How much property was a man assessed upon who paid $125 tax in Illinois in 1873? How much of this $125 was for State purposes? How much for school tax? If his property was assessed at ⅜ its real value, what was this man worth?

10. What school tax did a man in Illinois who was worth $20,000 pay in 1873, supposing his property to have been assessed at ⅓ its real value?

11. The total valuation of property in Wisconsin, as equalized by the State Board for 1874 was $421,285,359. The total tax levied for 1875 was $656,491.61. Of this tax $57,782.28 was for the two hospitals for the insane.

1st. According to the above, what was the tax on $1 assessment in Wisconsin for 1875?

2d. What tax would a man whose property was assessed at $5000 have had to pay in Wisconsin for 1875? How much would he have paid for the Hospitals for the Insane?

United States Revenue.

252. The expenses of the General Government* are provided for by a *Tax on Imported Goods,* and by the *Internal Revenue.*

253. ***Duties*** or ***Customs*** are taxes levied on imported articles, and are either *Specific* or *Ad Valorem.*

254. *Specific Duties* are duties levied on particular articles irrespective of their value. *Ad Valorem Duties* are duties levied on articles bought in foreign markets, and are estimated at a certain *per cent* on the net cost.

In transporting goods from one country to another there is more or less liability to loss, and consequently there are certain deductions made for such losses in cases in which specific duties are charged. The principal of these are *Draft,* an allowance by weight for waste; *Tare,* a deduction from gross weight, made for the weight of the box, bag, or other thing containing the goods; *Leakage,* a deduction made for actual loss through leakage from casks; *Breakage,* a loss by breakage from things imported in bottles.

In order to collect *Duties* or *Customs,* Congress determines on what articles, and at what rates they shall be charged, and the schedule embracing these facts is called a *Tariff.*† Congress also

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* That is, the government of the United States as a whole. These expenses are provided for by Congress, and embrace the salaries of U. S. officers, the expenses of the mail, army, and navy service, the improvement of navigation, expenses for national buildings, etc.

† From Tarifa, a fortress established by the Moors at the straits of Gibraltar, where they exacted duties from all vessels entering or leaving the Mediterranean Sea.

designates *Ports of Entry*, that is, ports where imported goods can be landed, and where *Custom-Houses* are built and officers of government kept to collect customs.

255. Internal Revenue is revenue derived from sale of public lands, from sale of postage stamps, from taxes on certain manufactures, as distilled and malt liquors, stamp duties, etc.

Laws regulating the internal revenue are called *Excise Laws*, in distinction from *Tariff Laws*, which regulate *Duties*.

Ex. 1. M. S. Smith, Detroit, Michigan, importer of watches, etc., received an invoice of 3 cases of Swiss watches, costing 22800 francs; duty 25%; cost of transportation 35 francs; commission to agent in Geneva 2½%. What was the cost in greenbacks, gold being 112½?

SOLUTION.

Net cost in Geneva,	22800	francs.
Duties paid in New York,	5700	"
Commission to Agent in Geneva,	570	"
Transportation,	35	"
Cost in francs,	29105	
	.193	
Cost in U. S. gold,	$5617.265	
	1.12½	
Cost in currency,	$6319.423+	

2. A dry goods importer received at Boston, from Liverpool, the following invoice:

650 *yd.*	Broadcloth	@	13*s.*
1246 *yd.*	Lace	@	2*s.*
1200 *yd.*	Coach Lace	@	11*d.*
1950 *yd.*	Ingrain Carpet	@	3*s.*
2560 *yd.*	Drugget	@	$^2/_4$.*

* $^2/_4$ is 2*s.* 4*d.* So 3*s.* 6*d.* is written $^3/_6$, 10*s.* 8*d.* $^{10}/_8$, etc.

The duty on the broadcloth, carpeting, and drugget was 30%, and on the laces 25%. What were the customs in currency,* gold being at 110? *Ans.*, $1868.25.

3. What is the duty, at 40%, on 110 chests of tea, each containing 67 *lbs.*, and invoiced at 90 *cts.* a pound, the tare being 9 *lbs.* a chest? *Ans.*, $2296.80.

4. What is the duty on 50 *hhd.* of molasses, 63 *gal.* each, at 20*c.* *pr.* *gal.*, leakage 3%? *Ans.*, $611.10.

5. What is the duty on 15 casks of Malaga wine, each holding 54 gallons, invoiced at 45 cents per gallon, allowing 2 per cent for leakage, the custom-house rate being 20 cents per gallon, and 25 per cent *ad valorem.*

Ans., $248.06.

On some articles both specific and *ad valorem* duties are charged, as is supposed in the last example.

6. A wholesale merchant in Boston imported 80 dozen bottles of Cologne water, invoiced at $5.25 per dozen. Allowing 5 per cent for breakage, and regarding a dozen bottles as equivalent to 2¼ gallons, what is the duty, the rate being $3 per gallon and 50 per cent *ad valorem*?

Ans., $826.50.

7. What is the duty on 20 *hhd.* sugar, invoiced at 1160 *lb.* each, tare being 10%, and the duty 3*c.* *pr.* *lb.*?

8. What is the duty on 3 lots linen hdkfs., which the appraiser classifies as follows:

1 lot	value	£34 4*s.* 6*d.*,	duty	35%.
1 lot	"	£14 12*s.* 6*d.*,	"	40%.
1 lot	"	£36 11*s.* 3*d.*,	"	40%.

How much currency will it take to pay the duty, gold being at 112½? *Ans.*, $177.68.

*** Our national paper money—greenbacks.**

9. How much currency, gold being at 111, will it take to pay the duties on 3 cases of French *Mousseline de laine,* containing 7563.8 meters, invoiced at .88 francs, the impost being 8*c. pr. yd.*, and 40% *ad valorem.*

Ans., $1304.92.

10. What is the duty on 5 *T.* 16 *cwt.* 3 *qrs.* 20 *lb.* of steel, invoiced at 25*c. pr. lb.*, the duty being 20% ?

Ans., $654.80.

In the U. S. Custom-Houses 112 *lb.* is called a hundred-weight, whence 28 *lb.* is a quarter; *i. e.*, the *Long Ton* is in use.

SECTION III.

BUSINESS CALCULATIONS IN PERCENTAGE INVOLVING THE ELEMENT OF TIME.

Interest.

256. ***Interest*** is money paid for the use of money.*

257. The ***Principal*** is the sum for the use of which *interest* is paid.

It will be seen that *Principal* corresponds to *Base*, as heretofore used, and *Interest* to Percentage. So also the *Amount* is the sum of principal and interest.

258. ***Simple Interest*** is interest which is considered as falling due only when the principal is paid, or when a partial payment is made. It is usually reckoned at a certain per cent per annum.

* As the basis on which interest is computed is always money, it is not deemed best to cumber the definition with any allusion to anything else.

259. *The Fundamental Formulæ* in simple interest are

1. $\boldsymbol{i = Prt}$, and ***2.*** $\boldsymbol{A = P + i = P + Prt = P(1 + rt)}$, in which P is the *Principal*, or base; r, the *Rate*;* t, the *Time* in years; i, the *Interest*, or percentage; and A the *Amount*.

DEMONSTRATION.—1. When a principal P is put at simple interest at rate r, for t years, the understanding is that the interest for one year is r times the principal, or Pr, and that for 2 years it is *twice* as much as for 1 year; for 3 *yr.*, 3 times as much; for $3\frac{1}{2}$ or $3\frac{1}{4}$ years, $3\frac{1}{2}$ or $3\frac{1}{4}$ times as much as for 1 *yr.* Hence the interest for 1 *yr.* being Pr, for t years it is t times as much, or Prt, and we have $i = Prt$.

2. To produce formula (2) we have but to remember that A, the amount, is the sum of principal and interest. Hence $A = P + i$, by definition. But as by (1) $i = Prt$, we have also $A = P + Prt$, and as $P + Prt$ is the same as $P(1 + rt)$, we have

$$A = P + i = P + Prt = P(1 + rt).$$

Ex. 1. What is the interest on \$345 for $3\frac{1}{2}$ years at 8%? What is the amount? Find the amount directly from the things given.

SOLUTION.—BY THE FORMULÆ.—

(1) $$i = Prt = 345 \times \tfrac{8}{100} \times 3\tfrac{1}{2} = \frac{\overset{69}{\cancel{345}} \times \overset{4}{\cancel{8}} \times 7}{\underset{\underset{5}{\cancel{25}}}{\cancel{100}} \times \cancel{2}} = 96.60.$$

(2) $$A = P(1 + rt) = 345(1 + \tfrac{8}{100} \times 3\tfrac{1}{2}) = 345\left(1 + \frac{8 \times 7}{100 \times 2}\right) = 345(1 + \tfrac{7}{25})$$
$$= 345 \times \tfrac{32}{25} = \frac{69 \times 32}{5} = 441.60.$$

* It will be observed that we use r to represent the *rate*, not the *rate per cent*. Using r to represent the *rate per cent*, (1) would be $i = \frac{Prt}{100}$, since the *rate* would be $\frac{r}{100}$. Thus if the *rate* per cent is 6, the *rate* is .06 or $\frac{6}{100}$.

BY ELEMENTARY ANALYSIS.—Since 8% means .08 of the principal, we have $\$345 \times .08 = 27.60$, as the interest for 1 *yr.* Again, as the understanding in simple interest is that the interest is in the direct ratio of the time, if the interest for 1 *yr.* is \$27.60, the interest for $3\frac{1}{2}$ *yr.* is $3\frac{1}{2}$ times \$27.60, or \$96.60.	\$345 .08 27.60 $3\frac{1}{2}$ 1380 8280 \$96.60

For practical purposes the amount is found by adding the interest to the principal.

To find the amount directly from the data (things given), we consider that as \$1 yields \$.08 in 1 year, in $3\frac{1}{2}$ years it yields \$.28; whence the *amount* of \$1 for $3\frac{1}{2}$ *yr.* at 8% is \$1.28. Hence the amount of \$345 is 345 times \$1.28. $345 \times 1.28 = 441.60$.

2. At what rate per cent does \$50 principal yield \$5.25 in 1 *yr.* 9 *mo.*

BY THE FORMULÆ.—We have given P, i, and t, to find r. Hence, using $i = Prt$, we have $r = \frac{i}{Pt} = \frac{5\frac{1}{4}}{50 \times 1\frac{3}{4}} = \frac{21}{50 \times 7} = \frac{3}{50} = .06$. Therefore the rate per cent is 6.

BY ELEMENTARY ANALYSIS. \$50 in 1 *yr.* 9 *mo.*, at 1%, yields \$.875 interest. Hence to yield \$5.25, will require a rate as many times 1% as \$.875 is contained times in \$5.25, or $5.25 \div .875$, which is 6.

3. In what time will \$1260 at 8%, amount to \$1617?

The quantities under consideration are P, r, A, and t; of which P, r, and A are given, to find t. (2) Contains these quantities. Hence taking $A = P + Prt$, and solving for t, we have

$$t = \frac{A-P}{Pr} = \frac{1617-1260}{1260 \times .08} = \frac{357}{100.8} = 3.541\tfrac{2}{3}.$$

Now $3.541\frac{2}{3}$ *yr.* = 3 *yr.* 6 *mo.* 15 *da.*

ELEMENTARY ANALYSIS.—The thing inquired about is *Time*, the given effect is interest. Hence find the interest of \$1260 at 8% for 1 year, and divide the given interest by it. Thus \$1260 at 8% gives \$100.80 in 1 *yr.* Hence to yield \$357 will require as many times 1 year as \$100.80 is contained times in \$357, etc.

4. What principal amounts to $232.50, in 2 *yr.* 11 *mo.*, at 10% ?

We have under consideration P, A, t, and r. Formula (2) contains these quantities; and as P is the thing sought, we solve $A = P(1+rt)$ for P, and have

$$P = \frac{A}{1+rt} = \frac{232.50}{1+\frac{1}{10}\times 2\frac{11}{12}} = \frac{232.50}{1+\frac{7}{24}} = \frac{232.5\times 24}{31} = 180.$$

ELEMENTARY ANALYSIS.—The given effect is *amount*, and the thing inquired about is *principal*. Hence, as $1 principal amounts to $1.29⅙ in 2 *yr.* 11 *mo.* at 10%, it will require as many times 1 dollar principal to amount to $232.50, as $1.29⅙ is contained times in $232.50, or $180.

260. PRACTICAL SUGGESTIONS.—1. *Let each example be solved both by means of the formulæ, and by the ordinary elementary analysis.*

2. *In using the formulæ it will be observed that there are always 4 things under consideration, 3 of which are given, and 1 is required. If either of the formulæ contains the 4 things under consideration use that one which does. If neither formula contains the 4 things to be considered in the problem, combine the two formulæ, eliminating the 5th quantity and use the formula thus produced.*

3. *Common fractions are generally more convenient in using the formulæ than decimals. Always cancel as much as possible.*

4. *The student should have much practice in reading the arithmetical rule from the formula.*

5. The amount being $167.65, the rate per cent 7, and the time 2 *yr.* 5 *mo.* 8 *da.*, what is the interest?

2 *yr.* 5 *mo.* 8 *da* $= 2$ *yr.* $+ \frac{5}{12}$ *yr.* $+ \frac{8}{360}$ *yr.* $= \frac{439}{180}$. As neither the formulæ (1) $i = Prt$, nor (2) $A = P + Prt$, contains the 4 quantities

A, r, t, and i, we combine the two, eliminating P. Thus from (1), $P=\frac{i}{rt}$, which substituted in $A=P(1+rt)$, gives $A=\frac{i}{rt}(1+rt)$; whence $i=\frac{Art}{1+rt}=\frac{167.65\times\frac{7}{100}\times\frac{439}{180}}{1+\frac{7}{100}\times\frac{439}{180}}=\frac{167.65\times 7\times 439}{18000+7\times 439}=\frac{515188.45}{21073}$ $=24.45-$.

261. ***Let the student write the rule for the case in which there are given the amount, rate, and time, to find the interest.***

$i=\frac{Art}{1+rt}$ gives as the rule, *Multiply together the amount, rate, and time, and divide the product by 1 + the product of the rate and time.*

6. In what time will \$419.84, yield \$41.28 interest, at 5%?

Formulæ (1), $i=Prt$, contains the things under consideration, of which t is the thing sought—the unknown quantity; hence $t=\frac{i}{Pr}=\frac{41.28}{419.84\times .05}=\frac{8256}{4198.4}=1.96646+$ *yr.* $=1$ *yr.* 11 *mo.* 18 *da.*

262. ***Let the student write a rule for this case.***

REMARK.—It will be an excellent exercise for the pupil to take the formula for any given problem and give the elementary analysis which justifies the formula. Thus to find the time when principal, interest, and rate are given, we have $t=\frac{i}{Pr}$. Now the principal multiplied by the rate gives the interest for 1 *yr.*, and dividing the interest for the number of years by the interest for 1 *yr.*, gives the number of years.

7. What is the rate per cent if \$450 amounts to \$687 in 6 *yr.* 7 *mo.* *Ans.*, 8.

From (2) we have $r=\frac{A-P}{Pt}=\frac{687-450}{450\times 6\frac{7}{12}}=\frac{237}{2962.5}=.08.$

263. *Let the student write the rule for this case, and give the elementary analysis which justifies the formula.*

Solve the following, observing the directions:

1st. Solve by the formulæ and also by the elementary analysis.

2d. Whenever a formula is used, give the arithmetical rule of which it is the expression, and justify the formula by analysis.

3d. In all cases seek the most elegant and expeditious methods of performing the mere numerical operations.

8. What is the interest of $240 for 1 *yr.* 9 *mo.* and 6 *da.*, at 7%?

9. What is the interest of $584 for 2 *yr.* 7 *mo.* and 27 *da.*, at 6%?

10. What principal yields $1.2924 interest in 1 *yr.* 6 *mo.*, at 6%?

11. What principal yields $76.095 interest in 3 *yr.* 8 *mo.* 15 *da.*, at 6%?

12. What principal yields $295.887 in 8 *yr.* 8 *mo.* 12 *da.*, at 10%?

13. At 8% in what time will $120 principal yield $22.08?

14. At 10% in what time will $150 principal amount to $178?

15. At 8% in what time will $420 principal amount to $441.84?

16. At what % will $40 yield $13.26 interest in 2 *yr.* 9 *mo.* 12 *da.*?

17. At what % will $125 yield $157.375 amount in 3 *yr.* 6 *mo.*?

18. At what % will \$36 amount to \$46.712 in 3 *yr.* 8 *mo.* 19 *da.*?

19. What is the amount of \$45 for 5 *yr.* 5 *mo.* and 15 *da.*, at 8%?

20. What is the amount of \$746.25 for 1 *yr.* 10 *mo.* and 12 *da.*, at 5%?

21. At what rate per cent will \$750 yield $\$224.33\frac{1}{3}$ interest in 3 *yr.* 8 *mo.* 26 *da.*?

22. In what time will \$750 yield $\$224.33\frac{1}{3}$ interest at 8%?

23. What principal will yield $\$224.33\frac{1}{3}$ interest in 3 *yr.* 8 *mo.* 26 *da.*, at 8%.

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264. Best method of Computing Interest.—There can be no question but that the best practical method of computing interest is to *Multiply the principal by the rate, and this product by the time expressed in years.*

Business men who have frequent occasion to compute interest always use tables. (See Elements of Arithmetic.)

24. What is the interest on \$750 for 3 *yr.* 8 *mo.* 26 *da.*, at 8%?

The form of operation in the margin will be found the most expedient for common use.

Interest for 1 *yr.* \$60. Hence for $3.73\frac{8}{9}$ *yr.* it is $3.73\frac{8}{9} \times 60$.

In this manner solve the following:

OPERATION.

30	26
12	8.86+
	$3.73\frac{8}{9}$
	60
Interest,	$\$224.33\frac{1}{3}$

25. What is the interest on \$250 from January 15, 1870, to June 10, 1872, at 6 per cent?

Ans., \$36.042—.

26. What is the interest on \$192.25 from February 20, 1871, to May 12, 1873, at 5 per cent?

27. What is the interest on \$370.12$\frac{1}{2}$ from September 5, 1872, to December 1, 1875, at 7 per cent?

28. What is the interest on \$500.18$\frac{3}{4}$ from October 28, 1874, to January 16, 1877, at 8 per cent?

29. What is the interest on \$734.62$\frac{1}{2}$ from November 13, 1875, to August 29, 1878, at 10 per cent?

Exact Interest.

265. The method of reckoning time by subtracting the dates and calling 30 days a month and 12 *mo.* a year, of course does not get the exact time. The exact time is found by reckoning the entire years, and the exact number of days in any fraction of a year which may be involved, and calling 1 *da.* $\frac{1}{365}$ of a year. (See ***273.***)

266. In practical affairs interest should usually be computed for 3 days more than the nominal time. These 3 days are called Days of Grace, and it is optional with the creditor whether he pays his note on the day on which it nominally falls due, or any time within the 3 succeeding days. An action at law can not be brought against him till the expiration of the 3 days.

Ex. 1. What is the exact interest on a note of \$345, dated June 12, 1874, payable Nov. 16, 1876, at 10%?

There are 2 entire years, and $18+31+31+30+31+16+3=160$ *da.* Hence the time is $2\frac{160}{365}$ *yr.*, or $\frac{178}{73}$. Hence we have $\$34.5 \times \frac{178}{73} = \$84.123+$.

2. What is the exact interest on $58.40, from Feb. 17, 1873, to Oct. 23, 1877, at 8%?

3. What is the amount due on a note July 24, 1875, which is dated Dec. 10, 1874, the principal being $500 and the rate per cent 7?

[If more exercises are needed the preceding set can be used, or the AUTHOR'S HAND-BOOK will furnish all that may be desired.]

Compound Interest.

267. Compound Interest is interest considered as falling due at regular intervals of time, and to be reckoned as increasing the interest-bearing debt from such times.

This method of reckoning interest allows interest on interest accrued, and hence the term *compound*, meaning *interest on interest.*

268. ***The Fundamental Formula*** for compound interest is $A = P(1+r)^n$, in which P represents the principal, r the rate, n the number of equal intervals of time at the end of which the interest is compounded,* and A the amount.

DEMONSTRATION.—To fix the thought we will consider that the interest is to be added to the principal at the end of each year, thus forming a new principal for the next year. Thus the interest on principal P for 1 year is Pr, and the amount $P+Pr$, or $P(1+r)$, which constitutes the principal for the 2d year. Now we see from the first operation that to find the amount for 1 *yr.* we simply multiply the principal for that year by $1+r$, hence at the end of the 2d year the amount is $P(1+r) \times (1+r)$, or $P(1+r)^2$. This being the principal for the 3d year, the amount at the end of the year

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* This is usually 1 *yr.*, or ½ a year; though sometimes interest is compounded quarterly. *When no interval is named, annually is always understood.*

is $P(1+r)^2 \times (1+r)$, or $P(1+r)^3$. Hence for n years the amount is $A = P(1+r)^n$.

Now it is evidently immaterial, so far as the formula is concerned, whether the interest is added to the principal, and a new principal formed at the end of each year, or at the end of each half, or quarter year, or month, or any other interval of time. Hence the formula is entirely general, n being the number of equal intervals.

Ex. 1. What is the compound interest on \$240 for 5 *yr.*, at 10% ?

First find the amount, $A = 240(1+.10)^5 = 240(1.1)^5 = 240 \times 1.61051 = 386.5224$. Hence as the interest is the amount minus the principal, we have $i =$ \$386.5224 − \$240 = \$146.5224.

2. Let the student show that the factor $(1+r)^n$ in the formula $A = P(1+r)^n$ is the amount of \$1 for the given rate and time, and hence that if we had a table giving the amount of \$1 for various rates and times, we could find the amount of any given principal by multiplying the corresponding tabulated amount by the given principal.

3. Show that the formula for compound interest is the same as for the last term of a series of which the first term is the principal P, the rate $1+r$, and the number of terms 1 more than the number of years, or intervals.

269. COMPOUND INTEREST TABLE.

Yrs.	2 per cent.	2½ per ct.	3 per ct.	3½ per ct.	4 per ct.	5 per ct.
1	1.0200 0000	1.025000	1.030000	1.035000	1.040000	1.050000
2	1.0404 0000	1.050625	1.060900	1.071225	1.081600	1.102500
3	1.0612 0800	1.076891	1.092727	1.108718	1.124864	1.157625
4	1.0824 3216	1.103813	1.125509	1.147523	1.169859	1.215506
5	1.1040 8080	1.131408	1.159274	1.187686	1.216653	1.276282
6	1.1261 6242	1.159693	1.194052	1.229255	1.265319	1.340096
7	1.1486 8567	1.188686	1.229874	1.272279	1.315932	1.407100
8	1.1716 5938	1.218403	1.266770	1.316809	1.368569	1.477455
9	1.1950 9257	1.248863	1.304773	1.362897	1.423312	1.551328
10	1.2189 9442	1.280085	1.343916	1.410599	1.480244	1.628895

Yrs.	6 per ct.	7 per ct.	8 per ct.	9 per ct.	10 per ct.	12 per ct.
1	1.060000	1.070000	1.080000	1.090000	1.100000	1.120000
2	1.123600	1.144900	1.166400	1.188100	1.210000	1.254400
3	1.191016	1.225043	1.259712	1.295029	1.331000	1.404908
4	1.262477	1.310796	1.360489	1.411582	1.464100	1.573519
5	1.338226	1.402552	1.469328	1.538624	1.610510	1.762342
6	1.418519	1.500730	1.586874	1.677100	1.771561	1.973822
7	1.503630	1.605781	1.713824	1.828039	1.948717	2.210681
8	1.593848	1.718186	1.850930	1.992563	2.143589	2.475963
9	1.689479	1.838459	1.999005	2.171893	2.357948	2.773078
10	1.790848	1.967151	2.158925	2.367364	2.593742	3.105848

4. What is the amount of $300 at compound interest for 8 *yr.*, at 7%? At 10%?

5. What is the compound interest on $550 at 3% for 9 *yr.*? For 7 *yr.*? For 3 *yr.*? For 15 *yr.*? For 24 *yr.*? For 18 *yr.*? For 25 *yr.*? For 21 *yr.*?

To find the amount of $550 for 21 *yr.*, at 3%, from the above table, consider that we have $A = P(1+r)^{21} = P(1+r)^{10} \times (1+r)^{10} \times (1+r) = 550 \times (1.343916)^2 \times 1.03$.

6. What is the amount of $3500 for 6 *yr.*, at 2%? At 4%? At 3½%? At 12%?

7. What is the compound interest on $500 at 8% per annum, compounded semi-annually for 3 *yr.*? What if compounded quarterly?

1. $A = P(1+r)^n = 500(1+.04)^6 = 500(1.04)^6 = 500 \times 1.265319$.
2. $A = P(1+r)^n = 500(1.02)^{12} = 500(1.02)^{10}(1.02)^2 = 500 \times 1.21899442 \times 1.0404$.

8. What is the compound interest on $75 for 5 *yr.*, at 7% per annum, compounded semi-annually? At 10% compounded semi-annually?

9. What is the amount at compound interest for 2 *yr.*, of $2500, on interest at 6%, compounded quarterly? What at 8% compounded quarterly? What at 5%?

10. What is the compound interest on \$450 at 7% for $2\frac{1}{2}$ *yr.*? For $3\frac{3}{4}$ *yr.*? For 3 *yr.* 10 *mo.*?

Find the amount of \$1 for the whole years, and add to it the required fractional part of the interest for the next year, as found by taking the difference between the amount for the entire years and the next amount. Thus,

Amount of \$1 at 7% for 3 *yr.*			**\$1.225043.**
Amount for 3 *yr.*	**\$1.225043**		
" " 4 *yr.*	**1.310796**		
Interest for 4th *yr.*	**.085753.**	**$\frac{3}{4}$ of .085753 =**	**.064315.**
Amount of \$1 for $3\frac{3}{4}$ *yr.* at 7%			**\$1.289358.**

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11. What principal yields as amount at compound interest at 5%, \$601.965, in 3 *yr.*?

Formula $A = P(1 + r)^n$, becomes $601{,}965 = P(1.05)^3$, whence $P = \frac{601.965}{(1.05)^3}$. The value of $(1.05)^3$ can be found by actual involution, or from the table.

12. What principal at compound interest for 2 *yr.* at 6% per annum, compounded semi-annually, yields \$42.67 interest?

Since $i = A - P$, we have $i = P(1+r)^n - P = P[(1+r)^n - 1]$, whence $P = \frac{i}{(1+r)^n - 1} = \frac{42.67}{(1.03)^4 - 1}$.

13. In what time will \$60 yield \$106.70 amount at compound interest at 7%?

From $A = P(1+r)^n$ we are to find n. Substituting the given values this becomes $106.70 = 60(1.07)^n$, or $(1.07)^n = \frac{106.70}{60} = 1.778333+$. Now the 7% column in the table gives the powers of 1.07. Thus

we find that $(1.07)^8 = 1.718186$, and $(1.07)^9 = 1.838459$. Hence the time is between 8 *yr.* and 9 *yr.* To find the fractional part of a year, we observe that the interest for the 9th year is $1.838459 - 1.718186 = .120273$. But $1.778333 - 1.718186 = .060147$. Hence $\frac{.060147}{.120273} = \frac{1}{2}$ nearly, is the fractional part of a year desired.

14. How long does it take any principal at 8% compound interest to double itself? How long at 9%? At 10%? At 12%? *Ans.*, At 12%, 6 *yr.* 40 *da.*+.

15. In what time will \$500 amount to \$669.112, at 6% compound interest?

16. In what time will \$150 yield \$75 interest, the interest being compounded yearly at 5%?

17. In what time will \$240 yield \$46.82 interest, the interest being compounded quarterly at 8% per annum?

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18. At what per cent will \$40 amount to \$50.50, at compound interest for 4 *yr.*?

From $A = P(1+r)^n$, we have $(1+r)^4 = \frac{50.50}{40}$, or $1+r = \sqrt[4]{1.2625} = \sqrt{1.123612} = 1.06$ very nearly. Hence $r = .06$.

19. At what per cent will \$250 become \$304.16, in 2 *yr.* 6 *mo.*, interest compounded semi-annually?

If more of this class of examples are desired, any of the preceding will furnish them.

Annual, Semi-Annual, and Quarterly Interest.

270. Contracts are often made in which it is agreed that the interest shall be paid annually, semi-annually, or even quarterly. This is, in fact, compounding the inter-

est thus often; but if the payments of interest are not made as they fall due, the general rule is that only simple interest can be collected, although the statutes of some of the States allow simple interest on *the deferred payments of interest.**

Ex. 1. Five years from date, for value received I promise to pay Abner Webb, or order, $600, with annual interest at 10%?

C. N. JONES.

TOLEDO, O., *Oct.* 5, 1868.

What is the amount of this note when due, only the first payment of interest having been made, remembering that the laws of Ohio allow simple interest on deferred payments of annual interest? *Ans.*, $876.64.

Remember the 3 days of grace.

2. Four years after date, for value received, I promise to pay C. H. Millen, or bearer, $175.50, with annual interest at 7%.

JACOB GILES.

ANN ARBOR, MICH., *Feb.* 15, 1871.

What was due at the maturity of this note, no payments having been made?

Michigan allows simple interest on deferred payments of annual interest.

* Annual interest can be collected at law like any other debt after it has become due, and if the creditor does not so proceed he is considered in some States as waiving that part of the contract which requires *annual* interest, while the laws of other States allow simple interest, as above stated. Separate notes are often given for annual interest, the principal note being without interest, and the interest notes bearing interest after due.

3. Allowing simple interest on deferred payments of annual interest, what was due on an 8% note of $500 bearing annual interest 5 *yr.* 8 *mo.* 25 *da.* after date, the first two payments of interest having been made as agreed?

4. Same as above on a 5% note for $350, payable 4 *yr.* 8 *mo.* 20 *da.* after date, 3 payments having been made?

5. Same as above on a 6% note for $200, due 3 *yr.* 6 *mo.* after date, no payments having been made?

6. What is the difference between the amounts of a 10% note for $500, running 5 *yr.* at annual interest as above, and at compound interest, no payments having been made in the former case?

Partial Payments.

271. It frequently happens that a debtor does not pay his note all at one time. In such a case, whatever is paid at any time is endorsed (credited) on the back of the note, and is called a ***Partial Payment*** (or simply, a payment).

There are several methods in more or less general use for computing interest on such notes. We give first the one known as the *United States Court Rule,* it having been adopted in the U. S. Courts, and in the courts of several of the States.

This rule is based upon these two principles:

1. *The principal can not be diminished until the accrued interest is paid;*

2. *Interest shall not draw interest.*

U. S. COURT RULE FOR COMPUTING INTEREST ON NOTES ON WHICH PARTIAL PAYMENTS HAVE BEEN MADE.

272. Rule.—I. *Compute the interest on the Principal from the date of the note to the time of the first payment. If this payment equals or exceeds this interest, find the amount and subtract the payment. Treat this remainder as a New Principal, and proceed to the next payment. Continue the process till the time of settlement is reached.*

II. *If any payment is less than the accrued interest, add such payment to the next, and treat the sum as one payment made at the latter date.*

273. NOTE.—*It should be generally understood that the U. S. Courts and most other courts, hold the common method of reckoning a day as $\frac{1}{360}$ of a year as illegal. Several of the States have statutes making a day $\frac{1}{365}$ of a year. The proper way, therefore, to reckon time in computing interest, is to reckon the calendar years, and the exact number of days as 365ths of a year. Attempts to collect claims with interest reckoned on the basis of a day as $\frac{1}{360}$ of a year, might endanger the entire claim in some of the States.*

MILWAUKEE, WIS., *May* 20, 1874.

Ex. 1. $1650.

One day after date, for value received, I promise to pay Israel Hall, or order, sixteen hundred and fifty dollars, with interest at 7%. PETER JONES.

On the back of this note were the following endorsements: Sept. 8, 1874, $45 ; Dec. 14, 1874, $30; Feb. 26, 1875, $50; July 5, 1875, $30; Nov. 14, 1875, $250.

What was due May 2, 1876?

OPERATION.

			Dates.	*Times.* Common Method.	Legal Method.
1874	5	20,	Date.		
1874	9	8,	\$45 Payment,	3 *m.* 18 *da.* = .3 *yr.*,	111 *da.*
1874	12	14,	\$30 Payment.	3 *m.* 6 *da.* = .2⅔ *yr.*,	97 *da.*
1875	2	26,	\$50 Payment,	2 *m.* 12 *da.* = .2 *yr.*,	74 *da.*
1875	7	5,	\$30 Payment,	4 *m.* 9 *da.* = .358⅓ *yr.*,	129 *da.*
1875	11	14,	\$250 Payment,	4 *m.* 9 *da.* = .358⅓ *yr.*,	132 *da.*
1876	5	2,	Settlement,	5 *m.* 18 *da.* = .4⅔ *yr.*,	171 *da.*

Common Method.			Legal Method.
Principal	\$1650.	. . . .	\$1650.
Interest to 1st Payment	34.65	Interest for $\frac{111}{365}$ *yr.*	35.124
Amount at 1st Payment	\$1684.65	. . . .	\$1685.124
1st Payment	45.00	. . . .	45.
New Principal. . . .	\$1639.65	. . . .	\$1640.124
Interest to 2d Payment*	30.61	Int. for $\frac{171}{365}$ *yr.*	53.787
Interest to 3d Payment	53.5619		
Amount at 3d Payment	\$1693.2119	. . . .	\$1693.911
Sum of 2d & 3d Pay'ts*	80.00	. . . .	80.00
New Principal . . .	\$1613.21196	. . . .	\$1613.911
Interest to 5th Payment†	80.9294	Interest for $\frac{261}{365}$ *yr.*	80.784
Amount at 5th Payment	\$1694.1414	. . . .	\$1694.695
Sum of 4th & 5th Pay'ts	280.00	. . . .	280.00
New Principal . . .	\$1414.1414	. . . .	\$1414.695
Interest to Settlement .	46.1952	Interest for $\frac{171}{365}$ *yr.*	46.394
Due at Settlement . .	\$1460.34–	. . . .	\$1461.09–

* Payment less than interest; hence we pass to the next payment, and deduct the sum of the payments.

† The fourth payment being but a little over ⅓ of a year from the third, and the interest for 1 *yr.* being \$112.92, we see at a glance that the fourth payment must be included in the fifth.

$950 PHILADELPHIA, *June* 26, 1874.

2. On the first day of January next, I promise to pay to John Reams, nine hundred and fifty dollars, with interest at 6%, for value received. SAM. VAN HORN.

This note is endorsed as follows: March 20, 1875, received $430; May 15, 1875, received $234.75. What was the balance due, June 1, 1876? *Ans.*, $353.02, or $352.97.

The following (computed at a day as $\frac{1}{365}$ of a year) will be found a convenient form for pupils to use in bringing results into class:

Principals.	*Time.*	*Interest.*	*Payments.*
$950	267 *da.*	$41.696 . . .	$430.
$561.696 . . .	56 *da.*	5.17 . . .	234.75
$332.116 . .	1 *yr.* 17 *da.*	20.855 . . .	$352.97 *Due.*

$840. NEW YORK, *Sept.* 8, 1872.

3. On demand, for value received, I promise to pay Thomas Trumble, or order, eight hundred and forty dollars, with interest at 7%. SIMEON BROOKFIELD.

On this note were endorsed the following payments: Oct. 1, 1873, received $44.56; Nov. 20, 1873, received $100.00; May 8, 1875, received $495.75. How much is due Dec. 29, 1875? *Ans.*, $364.56, or $364.41.

650\frac{50}{100}$. ST. LOUIS, MO., *July* 26, 1872.

4. Four years after date, I promise to pay John Wilson, or order, six hundred fifty and $\frac{50}{100}$ dollars, with interest at 6%. Value received. J. B. JACKSON.

On this note are the following endorsements: Received Jan. 20, 1873, $242.36; received Mar. 14, 1874, $144.90; received July 26, 1875, $267.30. How much remains due Sept. 8, 1876? *Ans.*, $74.49, or $74.58.

WASHINGTON, *January* 1, 1870.

$4000.

5. On demand, I promise to pay to James Wealthy, four thousand dollars, with interest at 10 per cent, for value received. JOHN READY.

This note was endorsed as follows:

July 1, 1871, received $300.

April 11, 1873, received $700.

Settlement was demanded, and full payment made, August 20, 1875. What was the balance then due?

Ans., 5254.44\frac{4}{9}$, or $5253.15+.

MERCHANT'S RULE.

274. It is a common practice with business men to treat obligations maturing and settled in a year or less, and upon which payments have been made, according to the following rule:

Find the amount of the principal from the date of the note to the time of settlement; find the amount of each payment from the time it was made to the time of settlement, and subtract their sum from the first result.

Ex. 1. On a 10% note for $460 running from Jan. 1 to Jan 1, the following endorsements were made:

Feb. 10, $120; May 6, $50; July 15, $100; Oct. 8, $80.

What was the balance due on settlement? (Exact interest, but no grace.) *Ans.*, $135.51.

2. For value received I promise to pay Morgan Brown, or bearer, Dec. 18, 1874, three hundred and fifty dollars, with interest at 7%. Dated May 13, 1874.

LEVI BORTEL.

Endorsements July 17, 1874, \$60; Sept. 20, 1874, \$80; Nov. 1, 1874, \$100.

What was due at maturity? (Exact interest, and grace.) *Ans.*, \$120.72.

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275. The following modification of the United States Court rule is in use in Connecticut:

CONNECTICUT RULE.

276. I. *When more than a year's unpaid interest has accrued at the time when any payment is made, the case is treated by the U. S. Court rule.*

II. *But when one or more payments are made* BEFORE *a year's unpaid interest has accrued, the sum of the amounts of these payments at the end of the year* * *is subtracted from the amount due on the note at that time, provided the amounts of the payments are sufficient to liquidate the interest then accrued. If any payment does not liquidate the interest accrued at the time it is made, it is simply carried forward without interest and added to the next payment, as in the U. S. Court rule.*

III. *When the final settlement is made before a year's unpaid interest has accrued, the several amounts are reckoned to the time of settlement.*

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* That is, the end of the year during which the interest has been accruing and unpaid.

Ex. 1. $900. DANBURY, *June* 1, 1868.

For value received I promise to pay Amos Harris, or order, nine hundred dollars, with interest at 6%.

PETER DULL.

Endorsements: June 16, 1869, $200.
Aug. 1, 1870, $160.
Nov. 16, 1870, $75.
Feb. 1, 1872, $220.

What was due August 1, 1872?

Dates.			*Payments.*	*Times.*
1868	6	1	Date	
1869	6	16	$200	1 *yr.* 0 *mo.* 15 *da.* = 1.04⅛ *yr.*
1870	8	1	$160	1 *yr.* 1 *mo.* 15 *da.* = 1.1¼ *yr.*
1870	11	16	$75	3 *mo.* 15 *da.* = .29⅙ *yr.*
1872	2	1	$220	1 *yr.* 2 *mo.* 15 *da.* = $1.2\frac{1}{12}$ *yr.*
1872	8	1	Settlement . .	6 *mo.* = ½ *yr.*

We see at a glance that the first two payments are to be treated by the U. S. Court rule. This leaves the balance due on the note

Aug. 1, 1870	$647.2968
1 *yr.* interest on this	38.8378
	$686.1346
Amount of $75 for 8½ *mo.*	78.1875
Balance due on note Aug. 1, 1871 . . .	$607.9471
Amount on this balance for 1 *yr.* . . .	36.4768
	$644.4239
Amount of $220 for 6 *mo.*	226.60
Balance due Aug. 1, 1872	$417.8239

2. What would have been due on the above if the third payment had been $35, and the fourth had been $260?

Ans., $420.83.

3. $1000. NEW HAVEN, *Apr.* 10, 1867.

For value received, I promise to pay to James Van Horn, or order, thirty days after date, one thousand dollars, with interest at 7%. MORGAN BROWN.

Endorsements: July 28, 1867, $500; Dec. 13, 1867, $8; Feb. 25, 1868, $12; May 7, 1868, $125; Oct. 3, 1868, $200; Mar. 15, 1869, $50.

What was due on the above note June 3, 1869? Compute both by the U. S. Court rule and by the Connecticut rule.

Ans., By U. S. Court rule $175.138; by Connecticut rule $173.14. Why does the latter rule give the less amount due?

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277. The ***Old Vermont Rule***, and the one which was in quite general use in the country half a century ago, was the same as the *Merchant's Rule* (**274**), without limitation to "notes running a year or less." By this rule the amount due on the note in Ex. 3, above, at the time of settlement would be $168.89.

This rule is, however, no longer in use in Vermont, the U. S. Court rule having taken its place, although Vermont, as well as several of the other States, has a special rule for notes drawing *Annual Interest*, which see (**280**).

Legal Rates.

278. ***Legal Interest*** is the rate per cent established by law as that which is to be implied in an interest-bearing obligation, in which the rate is not specified.

In Louisiana the legal rate is	5%
In the N. E. States, except Conn., and in N. C., Penn., Del., Md., Va., W. Va., Tenn., Ky., O., Mo., Miss., Ark., Ia., Ill., Ind., District of Columbia, and on debts due the United States	6%

In Conn., N. Y., N. J., S. C., Geo., Kan., Mich., Minn., and Wis.	7%
In Mo., Flor., and Texas	8%
In Col., Neb., Nev., Or., Cal., and Wash. Ter.	10%
In England and France	5%
In Canada, Nova Scotia, and Ireland	6%

279. *Usury* **is unlawful interest. Most of the States allow rates higher than those given above, when it is agreed upon between the parties and specified in the contract. Thus Penn. allows 7%; La., O., and N. C., up to 8%; Dist. Col., Ill., Ind., Ia., Ky., Mich., Miss., Wis., Mo., and Tenn., up to 10%; Minn., Or., Tex., and Va., to 12%; Neb., to 15%; and Ark., Arizona, Cal., Col., Dakota, Flor., Me., Mass., Nev., R. I., S. C., Utah, and Wash. Ter., any rate agreed upon.**

Partial Payments on Notes "with Annual Interest."

280. When partial payments are made on notes which bear *Annual Interest,* at other times than those at which the annual interest falls due, the method usually adopted is as follows:

Find the interest on the note for 1 year; and find also the amount of the payments made during the year, from the times they were severally made to the end of the year.

If the payments amount to more than the interest due, take their amount from the amount of the note, and make the remainder a new principal.

But if the amount of the payments does not equal the interest due, the principal remains unchanged, and the amount of the payments is taken from the interest, the remainder being treated as deferred interest.

Proceed in this manner with each year till the time of settlement, the last period being that from the time the last annual interest fell due to the time of settlement.

(*a.*) The times at which interest falls due, and to which interest on payments is reckoned, and at which the amounts of the payments are applied, are called *Rests.* Courts have allowed these rests to be made at Jan. 1 on such notes, instead of at the time at which annual interest fell due. In some cases Banks have been allowed to make these *Rests* quarterly.

(*b.*) In NEW HAMPSHIRE, if a payment made on a note bearing annual interest is less than the interest then due, it is carried forward and added to the next payment, *without interest,* and so on till the sum does exceed the interest, or to the time of settlement, when it is deducted. But when payments are made expressly on account of interest accruing, but not then due, they are applied when the interest falls due, *without interest* on so much of such payments as is necessary to cover the interest accruing.

Ex. 1. On a 10% note for $600, bearing annual interest, and dated June 12, 1873, there were the following endorsements:

June 12, 1874, $60;	Dec. 5, 1874, $100;
April 10, 1875, $50;	Nov. 4, 1876, $30;

What remained due Jan. 5, 1877?

Due June 12, 1874	$660.00
Paid June 12, 1874	60.00
Balance due June 12, 1874	$600.00
Interest for 1 year	60.00
Amount due June 12, 1875	$660.00
Amount of two payments made during this year, *i. e.*, $100 for 189 *da.*, and $50 for 63 *da.*, ($105.178 + $50.863)	$156.041
Balance due June 12, 1875	$503.959
Interest on this for 1 year	50.396
Amount on interest from June 12, 1876, to settlement Jan. 5, 1877, 207 *da.*	$554.355
Interest on above for 207 *da.*	31.286
Amount due Jan. 5, 1877	$585.792
Less amount of $30 payment for 62 *da.*	30.51
Balance due on settlement	$555.28

2. On a note for $1000, bearing 8% annual interest, and dated July 27, 1873, there were the following endorsements: Jan. 1, 1874, $50; Sept. 19, 1875, $150; July 27, 1876, $200; Feb. 3, 1877, $250. What was due Sept. 1, 1877? (Exact interest.) *Ans.*, $650.98

$500. CONCORD, N. H., *June* 7, 1873.

3. On demand, for value received, I promise to pay Enos Ames, or order, five hundred dollars, with annual interest at 6%. AMOS WHITE.

Endorsements: Feb. 10, 1874, $15; Aug. 15, 1874, $25; May 17, 1875, $150; Jan. 13, 1876, $20. What was due Oct. 18, 1876? What if the payments were made "on interest accruing"? *Ans.*, $379.13; $380.83

Partial Payments on Notes bearing Compound Interest.

281. *When partial payments are made on notes bearing compound interest, compute the interest on the several payments from the time they were made to the close of the respective years* in which they were made, and deduct the amount in each case from the amount of the note at the close of that year.*

Ex. 1. On a note for $500, bearing 5% interest compounded annually, dated June 7, 1873, there were the following endorsements: Nov. 15, 1873, $60; March 10, 1874, $80; Aug. 13, 1876, $150. What was due Oct. 17, 1877? *Ans.*, $291.67.

* Or other period at the end of which the interest is to be compounded.

282. PROBLEM.—*To find what each payment must be in order to discharge a given principal and interest in a given number of equal annual payments, by the U. S. Court Rule.*

SOLUTION.—Let b represent the principal (or base), r the rate, n the number of payments, and x one of the equal payments.

As the payments must exceed the interest falling due at the end of each year, we find the amount of b for 1 *yr.*, which is $b(1+r)$, and subtracting the payment x, have $b(1+r)-x$ as a new principal.

Multiplying this new principal by $(1+r)$, we have $b(1+r)^2-(1+r)x$ as its amount at the end of the 2d year. Subtracting the payment x, we have as a new principal $b(1+r)^2-(1+r)x-x$.

In like manner after the 3d payment we have $b(1+r)^3-(1+r)^2x-(1+r)x-x$.

After the 4th, $b(1+r)^4-(1+r)^3x-(1+r)^2x-(1+r)x-x$.

After the 5th, $b(1+r)^5-(1+r)^4x-(1+r)^3x-(1+r)^2x-(1+r)x-x$.

Hence after the nth payment we have $b(1+r)^n-(1+r)^{n-1}x-(1+r)^{n-2}x-$ etc. .... $(1+r)^2x-(1+r)x-x$.

But this equals 0 since the debt is discharged. Therefore the equation is $b(1+r)^n-(1+r)^{n-1}x-(1+r)^{n-2}x-$ etc. .... $(1+r)^3x-(1+r)^2x-(1+r)x-x=0$.

Transposing, changing the signs, and collecting the coefficients of x, this becomes $[(1+r)^{n-1}+(1+r)^{n-2}+$ etc. .... $(1+r)^3+(1+r)^2+(1+r)+1]x=b(1+r)^n$.

Dividing by the coefficient of x, we have

$$x=\frac{b(1+r)^n}{1+(1+r)+(1+r)^2+(1+r)^3+\text{etc.}..(1+r)^{n-2}+(1+r)^{n-1}}.$$

Now this denominator is evidently the sum of a geometrical series whose 1st term is 1, ratio $(1+r)$, number of terms n, and last term $(1+r)^{n-1}$. Whence substituting in the formula for the sum of a geometrical series, $s=\frac{lr-a}{r-1}$, we have $s=\frac{(1+r)(1+r)^{n-1}-1}{1+r-1}=\frac{(1+r)^n-1}{r}$ as the denominator of the value of x; hence $x=\frac{br(1+r)^n}{(1+r)^n-1}$ is the formula sought.

283. *Let the student translate this formula into a rule.*

Ex. 1. What must be one of the equal annual payments which will discharge a 10% note for $1200 in 5 years?

$$\therefore x = \frac{1200 \times \frac{1}{10}(1.1)^5}{(1.1)^5 - 1} = \frac{120 \times 1.61051}{.61051} = \$316.56-.$$

2. What must be one of the equal annual payments which will discharge a 7% note for $500 in 3 years?

Ans., $190.52+.

3. What must be one of the equal annual payments which will discharge an 8% note for $1000 in 4 years?

Discount.

284. ***Discount*** is a general term used by business men to signify any deduction made from a *nominal* price, or value. There are three principal uses of the term.

285. ***Commercial Discount*** is a deduction from the *nominal* price, or value, of an article.

286. ***Bank Discount*** is interest paid in advance, and for 3 days more than the nominal time. These 3 days are called *Days of Grace.*

This custom of allowing days of grace has become well nigh universal with reference to *Business Paper* (obligations for the payment of money). The general rule is that a suit at law cannot be instituted for the collection of any such paper until 3 days after its nominal maturity. Hence, in discounting such paper, it has become customary to compute the amount including these 3 days.

287. The ***Proceeds*** or ***Avails*** of a note given at bank is the amount which the bank pays for the note.

288. ***True Discount*** is a deduction made for the present payment of a sum of money due at some future time.

289. The ***Present Worth*** of a sum of money due at some future time, is a sum which, put at interest at a rate agreed upon, will in the given time amount to the sum due.

290. When the *Present Worth* of a note exceeds the *Face* of the note, this excess is called ***Premium.***

291. The difference between the nominal present value (as the *face* of a note) and the *Present Worth,* is the ***True Discount,*** or ***Premium,*** as the case may be.

Next after Simple Interest, Discount is the most important topic in Business Arithmetic.

Commercial Discount.

This subject presents no special difficulties, the problems being the same as those treated under *Profit and Loss* when the transaction involves loss.

Ex. 1. A dry-goods merchant, finding a piece of cloth which cost him \$3.75 per yard somewhat damaged, offered it for sale at 10% discount. What did he ask per yard for it?

2. A merchant sold some damaged cloth at \$3.37½ per yard, which was at a discount of 10% from the cost. What was the cost per yard?

3. A merchant sold cloth which cost him \$3.75 per yard, at \$3.37½. What per cent did he discount?

292. *Let the student state the Three Cases in Commercial Discount suggested by these three examples and give the rules for their solution.*

Bank Discount.

293. ***The Principle on which paper is discounted at bank*** is, Find what the obligation will yield at maturity, compute the interest on this amount from the time the paper is discounted to its maturity, and deduct this from the amount which the note will yield at maturity. This difference is the *Proceeds* (**287**), and the interest is the *Bank Discount* (**286**).

Bank Discount is always reckoned for the exact time in days, including the 3 days grace. Thus a note dated Dec. 4, and due in 2 *mo.* 15 *da.*, matures $27+31+19+3=80$ *da.* after date. In like manner, 3 *mo.* from July 14 is $17+31+30+14+3$ days. It is customary, however, in bank notes to express the time in days, as 60 *da.*, 30 *da.*, 90 *da.*, etc. A note for 60 *da.* matures in 63 *da.*; for 30, in 33 *da.*, etc. In view of this fact such dates are often written $^{60}/_{63}$ *da.*, $^{30}/_{33}$ *da.*, etc.

Ex. 1. Wishing to borrow some money at a bank for 60 days, they tell me they can accommodate me at 10%. I make a note for $500, and secure an endorser. What are the avails? What the discount?

Ans., Avails, $491.37; Discount, $8.63.

$\frac{63}{365}$ of $50 = $8.63. The *face* of this note was $500; the avails $491.37; the *discount* $8.63. The formulæ $i = brt$, and $D = b(1-rt)$ are those which apply to bank discount. Is i the face of the note, the avails, or the discount? What is b? What is D?

The *time* in bank discount is reckoned in days, and for a note

due in 60 *da.*, the time is written $^{60}/_{63}$, for 30 *da.* $^{30}/_{33}$, for 90 *da.* $^{90}/_{93}$. But the computation always includes the 3 days of grace. Banks always require an endorsement. (See Ex. 10.)

2. Wishing to obtain \$200 at bank for 90 *da.*, I have to pay 8%. For what must the note be made?

$$D = \$200,\ r = .08,\ t = \tfrac{93}{365},\ \text{to find } b.$$

Give the Elementary Analysis. \$1 face of note gives how much avails? (See *Elements, 264.*) *Ans.*, \$204.16.

3. How much shall I receive on my note for \$450 for $^{30}/_{33}$ *da.*, discounted at bank at 5%? At 10%? At 8% for $^{90}/_{93}$ *da.*? At 6% for $^{60}/_{63}$ *da.*?

For the last, $D = b(1 - rt) = b - brt = \$450 - 450 \times .06 \times \frac{63}{365} = \$445.34.$

4. In order to obtain \$350 at bank for $^{30}/_{33}$ *da.*, what must be the face of my note, discount being at 7%? What to get \$150 for $^{90}/_{93}$ *da.*, at 10%? To get \$750 for $^{60}/_{63}$ at 5%?

5. Avails \$300, time $^{30}/_{33}$ *da.*, rate .10. What was the face of note?

6. Face of note \$380, time $^{21}/_{24}$ *da.*, rate .08. What are the avails?

7. Face of note \$101.76, avails \$100, time $^{60}/_{63}$ *da.* What was the rate?

8. Face of note \$100, avails \$99.06, rate .10. What was the time?

9. Face of note \$1000, avails \$987, at 6%. What was the time?

10. $175. ANN ARBOR, MICH., *Feb.* 23, 1876.

Sixty days after date I promise to pay to the order of Jas. F. Royce, one hundred and seventy-five dollars at the ANN ARBOR SAVINGS BANK, for value received, with ten per cent interest after due.

EDWARD OLNEY.

The above being discounted at 10%, what were the proceeds?

This is a common form of note given when one borrows money at a bank. The one to whose order the note is made payable, as Jas. F. Royce in this note, writes his name on the back of the note, and becomes the *Endorser*. If the maker of the note does not pay it by the close of the third day of grace, a notice called a Protest is served on the endorser, and he becomes liable for its payment.

11. Having a 7% note for $360, dated March 12, 1874, due Feb. 12, 1877, on which is endorsed $50 June 15, 1875, I get it discounted at bank at 10% Dec. 11, 1876. What are the avails?

The balance due on this note Feb. $^{12}/_{15}$, 1877, is $381.74. This discounted at 10% for 66 *da.*, gives as avails $374.84.

294. The *Three Important Practical Problems* in Bank Discount are those illustrated by Exs. 10, 2, and 11. *Let the student state each and give a rule for its solution.*

12. What are the proceeds of the following note, discounted the day of its date; and when does it mature?

$100. ALBANY, *Oct.* 11, 1876.

Ninety days from date, for value received, I promise to pay to the order of John Smith, one hundred dollars, at the Albany City National Bank, interest 7%.

JOHN BROWN.

13. For what amount must I draw a note like the preceding if I wish to obtain $250?

14. I have John Smith's note for $280, dated July 25, 1874, bearing 8% interest, and due May 17, 1876. There is on it an endorsement of $100 Jan. 1, 1875. If I get it discounted at bank at 10%, March 15, 1876, what are the proceeds?

15. What are the proceeds of a 6% note for $500 bearing annual interest, dated July 23, 1873, due May 1, 1876, and discounted at bank, at 7%, Feb. 13, 1876, there being the following endorsements: July 23, 1874, $30; Jan. 14, 1875, $100; July 23, 1875, $40.

True, or Common Discount.

295. The Principle on which Common Discount is computed is, Find what the paper will yield at its maturity, and then find what sum put at interest at the rate of discount agreed upon, will yield the same amount at the time when the paper matures.

296. If the obligation is due at different times, the amount yielded at each of these times is treated as a separate obligation, and the sum of the present worths of all these amounts is the present worth of the entire obligation.

Ex. 1. I have a 7% note for $350 dated March 4, 1874, and due Nov. 12, 1876. July 25, 1875, Mr. B. proposes to buy the note at a discount of 10%. What must he pay me? *Ans.*, $367.99—.

By the principle of Common Discount (*295*), I find the amount of this note at maturity ($416.18), and then find what sum of money put at 10% interest July 25, 1875, will yield the same amount Nov. 12, 1876.

From July 25, 1875 to Nov. 15, 1876 is 1 *yr.* 113 *da.* Hence we have amount $416.18, time 1 *yr.* 113 *da*, rate .10, to find b (principal or base). The formula therefore is $A = b(1 + rt)$, or $b = \frac{A}{1+rt}$, which is the formula for True or Common Discount.

2. What is the present worth of a debt of $1860 due 4 *yr.* 9 *mo.* 18 *da.* hence, without interest, money being worth 5%? *Ans.*, $1500.

3. I have a 10% note for $280 dated Sept. 17, 1873, and due Feb. 6, 1876. May 23, 1875 Mr. C. proposes to buy it of me, discounting at 8%. What must he pay me?

4. Mr. C. gives me his note for $300, due 3 *yr.* hence, at 10%, and I sell it to Mr. B. the same day at 8% discount. What is its present worth? That is, what does B. pay me? *Ans.*, $314.52—.

Why is this note worth more than its face?

5. Mr. C. gives me his note for $300, due 3 *yr.* hence, at 8%, and I sell it the same day to Mr. B. at 10% discount. What is the present worth? That is, what does B. pay me? *Ans.*, $286.15+.

Why is this note worth less than its face?

6. Mr. C. gives me his note for $300, due 3 *yr.* hence, at 10% interest, which is all that money is worth. What is the present worth of the note on the day it is made? One year after its date what is its present worth? Two and one-half years after date?

7. Mr. C. gives me his note for \$300, due 3 *yr.* hence, without interest. What is it worth on the day it is given, money being worth 10%? What 1 *yr.* after date? What $2\frac{1}{2}$ *yr.* after date? What 3 *yr.* after date?

Why is such a note worth more as a longer time has elapsed after its date?

8. What is the difference between the bank discount of a note for \$500, running $^{90}/_{93}$ *da.*, at 10%, and the common discount for the same rate and time?

Ans., Bank Discount \$12.74; Common Discount \$12.42.

Observe that the amount of the common discount on interest at 10% for $^{90}/_{93}$ is exactly equal to the bank discount. Why is this?

Let the student trace the demonstration of this truth in the following: Calling D the bank discount, and D_1 the common discount, we have $D = brt$, and $D_1 = b - \frac{b}{1+rt} = \frac{brt}{1+rt}$. Now $\frac{brt}{1+rt} + \frac{br^2t^2}{1+rt} = \frac{brt(1+rt)}{1+rt} = brt$. Hence $D = D_1 + D_1rt$.

9. A note for \$800, dated March 18, 1867, and due in 3 *yr.*, with interest at 6%, payable annually, has the following endorsements: Oct. 24, 1868, \$150; Nov. 12, 1869, \$240. This note being discounted at bank, Jan. 3, 1870, at 10%, what were the proceeds? *Ans.*, \$533.71.

This note yields \$545.2094 March $^{18}/_{21}$, 1870; hence this is the amount to be discounted for 77 *da.*, *i. e.*, from Jan. 3 to March $^{18}/_{21}$, at 10%. In computing the amount the *time* is reckoned in the common way, so also in the next.

10. A note for \$1000, dated May 7, 1870, and bearing 7% annual interest, the principal payable in 4 equal annual installments, had on it the following endorsements: May 7, 1871, received \$70 interest due, and \$250

payment on the principal. Aug. 12, 1871, received \$400. What was the worth of this note Feb. 10, 1873, common discount at 10%, without grace, assuming that the payment (\$250) and interest falling due May 7, 1873, were promptly made, and the note settled at maturity?

Ans., \$395.96

May 7, 1871, after payment there was due . . .	\$750.00.
May 7, 1872, the note stood	\$381.89.
May 7, 1873, after payment there was due . . .	\$131.89.
May 7, 1874, the balance due was	\$141.12.

After the purchase the note yielded \$276.73 May 7, 1873, and \$141.12 May 7, 1874. The former sum is to be discounted for 86 *da.*, and the latter for 1 *yr.* 86 *da.**

N. B.—A single principle covers all such problems as the above, viz.: Find what the paper will yield, and when, and discount these amounts for their respective times.

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11. I have a claim which will yield \$600, 3 *yr.* hence. B. proposes to buy it at such a rate that he may receive 10% *annual interest* on his money. What must he pay?

Strictly speaking this is an impossibility. My claim does not bring in anything *annually*, but only at the end of 3 *yr.* But if 10% is the market value of money, *annual interest* is equivalent to *compound* interest. Hence I am to receive such a sum as put at compound interest for 3 *yr.* will amount to \$600. The problem then is, in compound interest, given the amount, rate, and time, to find the principal or base. Calling the principal or base b, we have

$$b = \frac{A}{(1+r)^3} = \frac{600}{(1+.10)^3} = \frac{600}{(1.1)^3} = \frac{600}{1.331} = 450.79-.$$

That this is a just solution in accordance with the conditions is evident, since at the end of the 1st year, although the note brings

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* Exact time. (See 273.)

in nothing, Mr. B. can borrow the interest \$45.079, at 10% annual or compound interest* for the remaining 2 *yr.*, when it will amount to \$54.545; at the end of the next year he can borrow his 10% annual interest on his investment, viz., \$45.079, which will amount to \$49.586 at the maturity of the principal note. Finally, when the note matures he receives \$600, which pays up his two notes for annual interest borrowed, and gives him \$45.079 as his 10% annual interest due at the maturity of the note.

\$54.545
49.586
45.079
450.79
\$600

ANOTHER VIEW may be taken of this case, based upon the rule allowing only simple interest on deferred payments of annual interest. See next examples.

12. What is the present worth of a note for \$500, bearing 7% annual interest, and due 4 *yr.* hence, discounted at date at 10% common discount, on the presumption that the annual interest will all be deferred till the maturity of the note?

Amount yielded by note at maturity (**270**) . \$654.70.
This discounted for 4 *yr.* at 10% is . . . . \$467.64+.

13. Same as above but discounting at 10% *annual interest*, *i. e.*, so as to allow the purchaser 10% annual interest on his money.

This may have two solutions according as we consider the 10% discount equivalent to compound interest, or as equivalent to deferred payments of annual interest, according to (**270**).

In the former case we should have $\frac{\$654.70}{(1.1)^4} = \$447.16$.

In case we treat the discount as *deferred annual interest*, we observe that \$1 at 10% *annual interest deferred* 4 *yr.*, amounts to \$1.46.† Hence the present worth on this hypothesis is $\frac{\$654.70}{1.46}$ $= \$448.42$.

* These are considered as the same in this discussion.

† \$1 gives 10*c.* annual interest. Hence we have 10*c.* on interest 3 *yr.* at 10% + 10*c.* on interest 2 *yr.* at 10% + 10*c.* on interest 1 *yr.* at 10% + 10*c.*, as the deferred annual interest on \$1. This is 46*c.*

14. To produce a general formula for discounting a sum due at a future time, allowing the purchaser deferred annual interest.

Let A represent the amount due and to be discounted, r the rate, t the time, and x the present worth. Now x is such a sum as put at annual interest and the interest deferred, that is, simple interest allowed on it, will amount to A in time t. The annual interest on x is rx, and of such annual interests there will be t, the last of which will bear no interest, the next to the last 1 *yr.* interest, the next preceding 2 *yr.*, the next preceding 3 *yr.*, the next 4 *yr.*, etc., to the first, which will bear $t-1$ *yr.* interest. Hence we have

The last year's annual interest. . . .	rx
Next to the last	$rx+rx\times r\times 1$
Second from the last	$rx+rx\times r\times 2$
Third from the last	$rx+rx\times r\times 3$
Fourth from the last	$rx+rx\times r\times 4$
Etc., etc., to the first.	
First year's annual interest.	$rx+rx\times r\times(t-1)$
Sum of interests . .	$trx+r^2(1+2+3+4+$ etc., to $t-1)x$

Hence the *amount* of x (the present worth) is $x+trx+r^2(1+2+3+4+$ etc., to $t-1)$, or $x[1+tr+r^2(1+2+3+4+$ etc., to $t-1)]$. But this is to equal the debt A. Hence we have

$$x[1+tr+r^2(1+2+3+4+\text{ etc., to }t-1)]=A;$$

whence $$x=\frac{A}{1+rt+r^2(1+2+3+4+\text{ etc., to }t-1)}$$

297. ***The Rule deduced from this is,*** *To 1 add the product of the rate and time, and the product of the square of the rate into the sum of the numbers 1, 2, 3, 4, etc., to 1 less than the number of years, and by this entire sum divide the amount of the debt.*

It will be observed that the divisor is the amount of \$1 on deferred annual interest for the time, at the rate at which the note is to be discounted.

15. What is the worth of a \$600 note with interest at 7%, dated June 6, 1873, and due October 10, 1876, on the 10th day of April 1874, allowing the purchaser 10% annual interest (deferred)?

Amount of note at maturity $\$740.4\dot{6}$.

From the time it is discounted to its maturity is $2\frac{1}{2}$ *yr.* Now \$1 at deferred annual interest for $2\frac{1}{2}$ *yr.* at 10% is $\$1+.10+.015+.10+.00\dot{5}+.05=\$1.27$.* Hence the worth of the note at the time of sale, under the conditions is $\frac{\$740.4\dot{6}}{1.27}=\$583.0445+$.

This is readily verified thus:

Paid for note	\$583.0445+
1 year's interest on investment at 10% . .	58.3044+
$1\frac{1}{2}$ year's interest on this interest	8.7456+
1 year's interest on investment	58.3044+
$\frac{1}{2}$ year's interest on this interest	2.9152+
$\frac{1}{2}$ year's interest on investment	29.1522+
Amount of note bought	\$740.4666

To apply the formula obtained above to this case we have but to notice that the series $1+2+3+4+$ etc., becomes $\frac{1}{2}+1\frac{1}{2}$, since the last deferred interest which draws interest draws it for $\frac{1}{2}$ *yr.*, and the next preceding for $1\frac{1}{2}$ *yr.* Hence we have $x=\frac{740.4\dot{6}}{1+\frac{25}{100}+\frac{1}{100}(\frac{1}{2}+1\frac{1}{2})}$ $=\frac{740.4\dot{6}}{1.27}$.

16. What is the worth of a \$1000 6% note dated June 7, 1874, and due 42 *mo.* from date, on the 7th of January, 1876, allowing the buyer 10% annual interest?

The amount of this note at maturity is $\$121\dot{0}$. This is to be discounted for $1\frac{11}{12}$ *yr.* Since this time is less than 2 *yr.*, the result will be the same whether we consider the purchaser to be allowed compound interest, or deferred annual interest, since there is no opportunity to get interest upon interest of interest, which marks the difference between compound interest and annual interest. Thus the amount of \$1 at compound interest for $1\frac{11}{12}$ *yr.* is $\$1.20\frac{1}{12}$, and the worth of the note is $\frac{\$1210}{1.20\frac{1}{12}}=\frac{\$14520}{14.41}=\$1007.63+$.

* The interest on \$1 at the end of the first year is 10*c.* $1\frac{1}{2}$ *yr.* interest on this is 1.5*c.* At the end of the 2d *yr.* another 10*c.* interest accrues, the interest on which for $\frac{1}{2}$ *yr.* is $\frac{1}{2}$*c.* Finally $\frac{1}{2}$ *yr.* interest on \$1 is 5*c.*

In like manner by the formula $x = \frac{A}{1 + rt + r^2(1 + 2 + 3 + \text{etc. to } t - 1)}$ we have $x = \frac{\$1210}{1 + \frac{1}{10} \times \frac{28}{12} + \frac{1}{100}\left(\frac{11}{12}\right)} = \frac{\$1210}{1.20\frac{1}{12}}$.

17. Having a note for $2500, due in 5 *yr.* from date, with simple interest at 7%, I get it discounted on the day it is made at 10% annual interest. What do I receive, considering the discount as deferred annual interest? What if annual interest be considered as equivalent to compound interest? What would be the present worth of the note at common discount? What at bank discount?

Answers in order, $2109.375; $2095.61; $2250; $1687.50.

18. A 5% note for $1000, due 20 *yr.* after date, has been running 10 *yr.*, when it is discounted at bank for the remaining time at 10% annual interest. What are the avails? *

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19. Mr. C. having taken a note for $600 bearing 5% annual interest, the principal payable in four equal annual installments, the whole secured by a good mortgage, steps into a Savings Bank, the same day, to get it discounted. They discount it so as to realize 10% on their money. What are the proceeds to Mr. C.?

SOLUTION.—Let x represent the proceeds. The note yields $180 at the end of 1st *yr.*; $172.50 at the end of the 2d *yr.*; $165. at the end of the 3d *yr.*; and $157.50 at the end of the 4th *yr.*

Now at the end of each year the bank is to have 10% on its investment *for that* year, and the remainder of the amount

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* Of course such a transaction is purely fictitious, since such a note would not be discounted at bank. Banks do not usually discount for more than $^{90}/_{93}$ *da.*, except in cases like the following.

which the note yields for that year can be applied to extinguish the investment. Observe that this is equivalent to finding the *amount* of the bank's investment for the year at 10%, and from this amount deducting the yield of the note for that year. The remainder is the bank's investment in the paper for the next year. To find the amount of any sum for 1 *yr.* we multiply by $1+r$, in this case by 1.1. The computation is as follows:

Amount of bank's investment for 1st *yr.* . . .	$1.1x$
Amount yielded by note at end of 1st *yr.* . . .	180
Bank's investment for 2d *yr.*	$1.1x-180$
Amount of bank's investment for 2d *yr.* . . .	$(1.1)^2x-198$
Amount yielded by note at end of 2d *yr.* . . .	172.50
Bank's investment for 3d *yr.*	$(1.1)^2x-370.50$
Amount of bank's investment for 3d *yr.* . . .	$(1.1)^3x-407.55$
Amount yielded by note at end of 3d *yr.* . . .	165
Bank's investment for 4th *yr.*	$(1.1)^3x-572.55$
Amount of bank's investment for 4th *yr.* . . .	$(1.1)^4x-629.805$
Amount yielded by note at end of 4th *yr.* . . .	157.50
Amount of bank's investment for 5th *yr.* . . .	$(1.1)^4x-787.305$

But this should be 0 since the note is paid up. Hence we have $(1.1)^4x-787.305=0$; or $(1.1)^4x=787.305$. Whence

$$x=\frac{787.305}{(1.1)^4}=\frac{787.305}{1.4641}=537.74.$$

This result is readily verified as follows:

Amount paid for note	\$537.74
Amount of investment for 1 year	591.514
Payment on note at end of 1st year	180.
Bank's investment for 2d year	\$411.514
Amount of investment for 2d year	452.665
Payment on note at end of 2d year	172.50
Bank's investment for 3d year	\$280.165
Amount of investment for 3d year	308.181
Payment on note at end of 3d year	165
Bank's investment for 4th year	\$143.181
Amount of investment for 4th year	157.50
Payment on note at end of 4th year	157.50
Bank's investment for 5th year	00.00

20. *To produce a formula for finding the present worth of a note bearing annual interest and payable in annual installments, the purchaser being allowed annual interest on his money.*

SOLUTION.—Let $a, b, c, d, \ldots . l, m, n, s,$ represent the sums yielded by the note at the end of each year successively (that is, the annual interest plus the payment on the principal). Let r be the rate, t the number of years, and x the present worth.

Then proceeding as under Ex. 19, we have

Bank's investment for 1st year .	x
Bank's investment for 2d year .	$(1+r)x-a$
For 3d year	$(1+r)^2x-(1+r)a-b$
For 4th year	$(1+r)^3x-(1+r)^2a-(1+r)b-c$

The law is now evident and we can write the formula for the investment for the $(t+1)$th year, which is $(1+r)^t x-(1+r)^{t-1}a-(1+r)^{t-2}b-$ etc., to $(1+r)^3l-(1+r)^2m-(1+r)n-s$. But this must be 0 since the note is paid up. Hence we have $(1+r)^tx-(1+r)^{t-1}a-(1+r)^{t-2}b-$ etc., to $(1+r)^3l-(1+r)^2m-(1+r)n-s=0$. This gives

$$x=\frac{(1+r)^{t-1}a+(1+r)^{t-2}b+\text{ etc., to }(1+r)^3l+(1+r)^2m+(1+r)n+s}{(1+r)^t}$$

298. From this formula a general rule for similar cases is readily written. The first payment (including the annual interest) is to be multiplied by 1 + *the rate* raised to a power whose index is 1 less than the number of payments; the second by the next lower power of 1 + *the rate,* etc.; and the sum of these products divided by 1 + *the rate* raised to a power whose index is the number of payments.

21. What is the present worth of a note for \$2500, bearing 8% annual interest, and payable in three annual installments of \$500, \$800, and \$1200 respectively, the purchaser being allowed 10% annual interest on his money? *Ans.,* \$2403.46—.

22. A certain railroad borrowed \$110,000 at 6%, agreeing to pay principal and interest in 6 equal semi-annual installments. But before the first month had expired the contract was changed so that the payments were to be equal *monthly* payments, the creditor allowing the road 6% simple interest on the monthly payments of each 6 months, from the time each was made to the close of that half year. What was one of the monthly payments?

In the first place we find by (*281*), one of the payments according to the original contract. Thus,

$$x = \frac{br(1+r)^n}{(1+r)^n - 1} = \frac{110{,}000 \times .03\,(1.03)^6}{(103)^6 - 1} = 20305.75+.$$

Secondly, we find what \$1 paid at the end of each month amounts to at the close of 6 *mo.* The several interests are $\frac{5}{2}+\frac{4}{2}+\frac{3}{2}+\frac{2}{2}+\frac{1}{2}$ $= 7\frac{1}{2}$ cents. Whence the amount of \$1 paid at the end of each month according to the new contract is \$6.075. And, finally, the monthly payment must be $\dfrac{\$20305.75+}{6.075} = \$3342.51.$

23. Same as 22 except that according to the original contract the interest on unpaid principal is to be paid semi-annually, and the principal in 6 equal semi-annual payments.

Solution.—Letting P represent the principal, and r the rate, there falls due the following sums, at the end of each 6 *mo.* respectively: $\frac{1}{6}P+\frac{1}{2}rP$, $\frac{1}{6}P+\frac{5}{12}rP$, $\frac{1}{6}P+\frac{4}{12}rP$, $\frac{1}{6}P+\frac{3}{12}rP$, $\frac{1}{6}P+\frac{2}{12}rP$, $\frac{1}{6}P+\frac{1}{12}rP$. Applying to the case the principle of the Merchant's rule, or the old Vermont rule, as the most equitable under the circumstances, we compute the interest on the first of these sums for $2\frac{1}{2}$ *yr.*, on the 2d for 2 *yr.*, on the 3d for $1\frac{1}{2}$ *yr.*, on the 4th for 1 *yr.*, on the 5th for $\frac{1}{2}$ *yr.*, and include the 6th without interest. This gives us the *Amount of the obligation at maturity*, $\dfrac{12+21r}{12}P$*

* Sum of payments.

$+\left[\frac{5}{2}r\left(\frac{2+6r}{12}\right)P+\frac{4}{2}r\left(\frac{2+5r}{12}\right)P+\frac{3}{2}r\left(\frac{2+4r}{12}\right)P+\frac{2}{2}r\left(\frac{2+3r}{12}\right)P+\right.$
$\left.\frac{1}{2}r\left(\frac{2+2r}{12}\right)P\ \right]^{*}=\frac{24+72r+70r^2}{24}P.$

Again, letting x represent one of the equal monthly payments, and allowing 5 *mo.* interest on the 1st, 4 on the 2d, 3 on the 3d, 2 on the 4th, 1 on the 5th, and none on the 6th, the *amount* of the payments for each six months, is found to be $\frac{24+5r}{4}x$.

On these amounts we are to reckon interest up to the time of settlement, that is, $2\frac{1}{2}$ *yr.* interest on the 1st, 2 on the 2d, $1\frac{1}{2}$ on the 3d, etc. This gives $\frac{5}{2}r\left(\frac{24+5r}{4}\right)x+\frac{4}{2}r\left(\frac{24+5r}{4}\right)x+\frac{3}{2}r\left(\frac{24+5r}{4}\right)x+\frac{2}{2}r\left(\frac{24+5r}{4}\right)x+\frac{1}{2}r\left(\frac{24+5r}{4}\right)x+\frac{24+5r}{4}x=\frac{15r+2}{2}\times\frac{24+5r}{4}x=\frac{48+370r+75r^2}{8}x$. To this add the sum of the six amounts, each $\frac{24+5r}{4}$, and we have as the total amount of the monthly payments up to the time of settlement $\frac{336+430r+75r^2}{8}x$. But this should equal the amount of the obligation. Hence we have

$$\frac{336+330r+75r^2}{8}x=\frac{24+72r+70r^2}{24}P;$$

whence $\qquad x=\frac{24+72r+70r^2}{1008+990r+225r^2}P=\$2942.23.$

one of the equal monthly payments.

Stocks and Bonds.

299. A ***Company*** is an association of persons for transacting business.

Business Companies are of two general classes, *incorporated* and *unincorporated.* The former are spoken of

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* Interest accrued on payments.

as *Corporations*,* and the latter as *Firms, Houses*, or *Partnerships*.

300. A ***Business Corporation*** is an association authorized by special or general law to transact certain business, under a specified name.

301. A ***Firm***, or ***House***, is an association bound to each other by mutual articles of agreement for the transaction of certain business.

Many business *firms* are also *corporations*.

The liabilities and prerogatives of incorporate and unincorporate companies are in many respects different, and are regulated by law. A member of a business corporation is usually liable for its debts only to the amount of the capital stock he owns, while a member of a Firm or House is liable to the full extent of his property, or the indebtedness of the firm.

302. ***Capital Stock***, or *Joint Stock*, is the amount of money paid, together with that subscribed but not yet paid in, for the purpose of carrying on the business of the company, or corporation.

When a company is formed for any particular purpose, as for example, to build and run a railroad, an estimate is made of the amount of money which will probably be needed to do the work and carry on the business, and this amount is divided up into what are called *Shares*, a share usually being $100. Then subscription-books are opened, and all who will are invited to "*take stock*," *i. e.*, to subscribe. When one subscribes for a certain number of shares, he agrees to pay in for the uses of the company the amount for which he subscribes, or such part of it as may be needed, when called for, and he receives from the company a *Certificate of Stock* which certifies that he owns so much stock in the company.

* The term *Corporation* has also a more extended sense, as when cities or villages are called corporations, and especially in Europe, where an office rank, or title, as king, bishop, etc., constitutes a "*Corporation sole*," entirely independent of the person occupying it.

303. ***Stocks*** are the certificates of a corporation, signed by the proper officers, and showing that the holder owns so many shares in the capital stock of the company.

304. Any one who owns stocks is a *Stockholder* in the company.

The ***Directors*** of a company are members elected by the stockholders for the general oversight and direction of the business, to which the President, Secretary, and other special officers give more immediate attention.

In electing directors, each stockholder is entitled to one vote for every share he owns.

When the railroad is built, or other business for which the company is organized, is properly under way, it is presumed that there will be money made by the road or business.

305. The ***Gross Earnings*** of a company is the total amount of money, or its equivalent, received in the transaction of its business.

306. The ***Net Earnings*** of a company is the amount that is left after deducting from the gross earnings the expenses of conducting its business, losses, and accrued interest upon its bonds or other obligations.

The net earnings of a company are its profits, and are to be divided among the stockholders in proportion to the amount of stock which each one owns, unless otherwise determined by the directors. The net earnings may be devoted to extending the business, if the directors so determine.

But business corporations are liable to losses in business, as well as are private individuals.

307. An ***Assessment*** is a sum required of stockholders to meet the losses, or expenses of the company.

In such enterprises as the construction of railroads, or other business which requires a gradual expenditure of money for considerable time before it is in condition to earn anything, it is not customary to require the payment in full on the stock taken; but assessments are made from time to time, as the work progresses. Such assessments are sometimes called *Installments*. No stockholder, however, can be called upon in this way for more than the amount of stocks which he owns.

Companies, like individuals, often find that their business requires more capital than they possess, and hence they resort to borrowing.

308. The ***Bond*** of a corporation is its certificate of indebtedness, signed by the proper officers, and given under the corporate seal.

Such bonds are the notes of the corporation, and are secured by mortgage upon its property. These bonds, like other notes, are made payable at a certain time, and bear a specified rate of interest. The bonds of a corporation are usually considered a surer investment than the stocks, since *dividends* on the latter are made only when the business receipts exceed the expenditures, whereas the *interest* on the former is due at the times nominated whether there be profits in the business or not; and the mortgage securing the bond may be foreclosed like other mortgages. Nevertheless, in some very lucrative business the stocks may be more valuable than the bonds.

309. ***Government Bonds*** are certificates of indebtedness issued by the government; as by the United States, or State government, by a county, city, school district, or other government corporation.

Such bonds are usually made payable at a certain time, and bear a specified rate of interest.

The occasions for these bonds are such as when a school district wishes to build a fine house, but does not want to increase the taxes sufficiently to pay for it in a single year, but prefers to distribute the payment over several years; and in like manner, when

a county or State is called upon to expend more money than it is deemed expedient to raise by immediate taxation. But the fruitful cause of such government indebtedness is *war*. In consequence of our late war, the indebtedness of the United States Government ran up from $88,995,810 in 1861, to $2,639,382,572 in 1867. In like manner, England and France have accumulated enormous debts. The national debt of England is mainly consolidated into a form of obligations which it is never expected that the government will pay, but which draw annual interest at 3%.

310. Consols are English government stocks. ***Rentes**** are French government stocks.

(*a*) *Consols* are properly but perpetual 3% *annuities*, as the principal of the debt is not presumed to be payable. The consolidated debt of England is £731,413,523.

(*b*) French Government Stocks, *Rentes*, bear various rates of interest. At the breaking out of the Franco-Prussian war, 1870, the debt of France was 11,516,469,221 *francs*, nearly all of which was 3% *Rentes*. There were 4 principal loans made in consequence of the war, viz., Aug. 1870, 1,328,282,839 *fr.* at 3%, bonds sold at 60 *fr.* 60*c.*; † Oct. 1870, 252,500,000 *fr.* at 6%, bonds sold at 85; June 1871, 2,777,952,800 *fr.* at 5%, bonds sold at 82.50; Jan. 1872, 3,000,000,000 *fr.* bonds sold at 84.50.

(*c*) The total interest-bearing debt of the United States Nov. 30, 1875 was $1,708,251,300. The total debt was $2,242,946,771.29.

Of the interest-bearing debt $1,033,866,550 is in 6% bonds, and $660,384,750 in 5%. In the technical language of commercial quotations, "U. S. 6's" are 6% bonds payable at a fixed time. Of these $18,415,000 are payable Dec. 31, 1880, and $945,000, July 1, 1881. "U. S. 6's 5-20," are 6% bonds payable any time from 5 to 20 years from their date, at the option of the government. "U. S. 10-40," are 5% bonds payable any time from 10 to 40 years after their date, at the option of the government. These were issued in

* Pronounced "*rahnts*." In strict language the term *Rentes* applies only to the *interest*, the principal—the debt itself—being called *Nominal Capital*.

† This means that a 100 *fr.* bond sold for 60 francs and 60 centimes, and others at this rate.

1864. "U. S. 6's 5-20 of '84," are 6% 5-20's issued in 1864, and hence due in 1884, etc.

(*d*) The interest on all these bonds is payable in gold, on 6% 5-20's, and 5% 10-40's semi-annually; the funded debt "5's of '81," bear 5% in gold, payable quarterly. The funded loan constitutes about $\frac{1}{4}$ of the interest-bearing debt.

311. A ***Coupon*** is a certificate of interest attached to a bond, which on the payment of the interest is cut off and delivered to the payor.

Stocks and Bonds are bought and sold in the market just as wheat or cotton, and the prices fluctuate according to prosperity of business, the plenty or scarcity of money, and many other circumstances.

312. ***Stock-jobbing*** is the business of buying and selling Stocks and Bonds, with a view to speculation.

313. The ***Par Value*** of stock is the face of the certificate or bond.

Thus the par value of 50 shares* is $5000, irrespective of what they may have cost the holder. The par value of a $1000 bond is $1000, etc.

314. ***Premium, Discount,*** *and* ***Brokerage*** *are always reckoned on the* ***Par Value.***

315. When stocks sell for more than their par value they are said to be at a ***Premium;*** when for less, they are at ***Discount.***

Stocks selling for 5% premium are said to sell at 105, *i. e.*, $1 stock brings $1.05, or it sells for $105 per share. Stocks at 87, are at 13% discount; *i. e.*, 1 share can be bought for $87. Stock at par is quoted at 100.

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* A share is always $100 unless otherwise stated.

316. The ***Market Value*** of stock is the price per share at which it can be bought.

Ex. 1. I find in my daily paper U. S. 5's of '81 quoted at $119\frac{7}{8}$, and gold at $112\frac{7}{8}$. Currency being worth 10%, what interest shall I receive if I invest greenbacks in these bonds.

SOLUTION.—An investment of $119\frac{7}{8}$, or 119.875 would yield me \$5 in gold per annum, *i. e.*, \$1.25 at the end of each 3 *mo.* (***310***, *d*). Allowing 9 *mo.* interest at 10% on the first payment, 6 *mo.* on the second, and 3 *mo.* on the third, the \$5 amounts to \$5.1875 in gold at the end of the year. This at $112\frac{7}{8}$ is equivalent to \$5.855390625 in currency. Hence an investment of \$119.875 in currency yields me in 1 year \$5.855390625, or 4.89%, or practically about 5%.

2. How much currency must I send to a New York broker in order to secure \$5000 U. S. 5-20's quoted at $117\frac{3}{4}$, brokerage $\frac{1}{4}$% ?

As the brokerage is reckoned on the par value of the bonds, \$100 in bonds will cost me $\$117\frac{3}{4} + \$\frac{1}{4}$ for brokerage, *i. e.*, \$118. Hence 50 hundred dollars in bonds will cost $\$118 \times 50 = \$5900$.

3. I find Tennessee State bonds (new) quoted at 44. What will \$20000 of these bonds cost me, with brokerage at $\frac{1}{8}$% ? *Ans.*, \$8825.

4. "Michigan 7's," *i. e.*, State bonds bearing 7% annual interest in currency are quoted at 109. What per cent do they yield on an investment? *Ans.*, 6.42% +.

5. The French 5% loan of 1872 was sold at $84\frac{1}{2}$. What per cent did the government pay on the money actually received?* *Ans.*, $5\frac{155}{169}$%, or nearly 6%.

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* This assumes that the $84\frac{1}{2}$ was paid to the government at the time of purchase of bonds. This was not the fact. Thus a person taking 100 francs of the loan paid $14\frac{1}{2}$ francs down and the remainder, 70 francs, in 20 equal monthly installments, without interest.

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6. The German 5% loan of 1871 was sold at $96\frac{1}{2}$. What per cent did the government pay on the money received?

Ans., $5\frac{35}{193}$%, or nearly $5\frac{1}{5}$%.

7. The German loan of 1871 was to be redeemed at par in 4 years (nearly). If the French loan of 1872 ran 10 years, which was the better loan, the German of 1871, or the French of 1872, reckoning the annual interest as paid by the government as equivalent to compound interest?

The rates were, French to German as $\frac{162.8895}{84\frac{1}{4}}$ to $\frac{121.5506}{96\frac{1}{2}}$, or 193– to 125+.

8. Western Union Telegraph stock is quoted at $77\frac{3}{8}$. How much will 100 shares cost me, if I send to a New York broker to buy it for me, and pay him $\frac{1}{5}$% brokerage, the exchange costing me $\frac{1}{10}$%?

100 shares at $77\frac{3}{8}$ =	**$7737.50**
Brokerage on $10000 at $\frac{1}{5}$%	**20.00**
	$7757.50

Hence I wish to send a draft for this amount. This at $\frac{1}{10}$% exchange will cost me $7757.50 ÷ .999, which with 2c. for revenue stamp will make the total cost $7765.28+.

9. Michigan Central R. R. stock is quoted to-day at $60\frac{1}{4}$ in New York, and New York exchange at $\frac{1}{10}$ premium in Detroit. How much of this stock can I buy in New York for $3384.40 currency in hand in Detroit, paying $\frac{1}{8}$% brokerage in New York?

Deducting 2c. for the revenue stamp, I have $3384.38 with which to buy a draft on New York. The exchange being at a premium of $\frac{1}{10}$%, $1.001 of currency will buy only $1 New York draft. Hence I can buy a draft of $3384.38 ÷ 1.001 = $3380.999, or $3381. Now every share will cost me $$60\frac{1}{4}$ + the brokerage on a share, *i. e.*, $\frac{1}{8}$% of $100 = .125. Hence a share will cost me in New York $60.25 + $.125 = $60.375; and I can buy 3381 ÷ 60.375, or 56 shares.

10. If the Georgia Railroad and Banking Company declares a semi-annual dividend of $3\frac{1}{2}$ per cent, what sum will a stockholder receive during the year who owns 175 shares? *Ans.*, $1225.

11. If a dividend of $12\frac{1}{2}$ per cent is declared upon the profits of the Graniteville Manufacturing Company, what sum will a stockholder receive who owns 15 shares, the par value of a share being $500? *Ans.*, $937.50.

12. If a railroad declares a dividend of 8 per cent, what sum will a stockholder receive who owns 54 shares, the par value being $100 per share? *Ans.*, $432.

13. What % shall I receive on my money if I buy stocks at $110\frac{1}{2}$ which pay 10% dividends?

14. What % shall I receive on an investment in stocks at 85 which pay a semi-annual dividend of 4%, reckoning money worth 10% annual interest?

An investment of $85, yields $4 + the interest on $4 at 10% for 6 *mos.*, + $4, or $8.20. Hence the % I receive is $820 \div 85 = 9\frac{11}{17}$.

15. At what must I buy stocks which pay 7% annual dividends to realize 10% on my investment?

Ans., At 70.

16. The net earnings of a company with a capital of $480000 are $35000; reserving $3000 as surplus,* what per cent dividend can they declare? *Ans.*, $6\frac{2}{3}$%.

17. How much money must a man living in Louisville, Ky., invest in order to buy in New York 50 shares N. J. R. R. stock, quoted at $107\frac{3}{4}$, brokerage $\frac{1}{4}$%, New York exchange $\frac{1}{8}$% premium? *Ans.*, $5410.82.

Remember the revenue stamp on the draft.

* Surplus is a portion of the net earnings reserved for meeting any contingencies which may arise, for extending the business, or other purposes which may need more capital, or income.

18. A man living in Milwaukee has $15000 to invest. His N. Y. advices lead him to invest in Illinois Central at 98$\frac{3}{4}$. How many entire shares can he secure, and how much of the $15000 will remain, the N. Y. broker charging him $\frac{1}{4}$%, and his N. Y. draft costing him $\frac{1}{8}$% in Milwaukee?

After securing his stock he will have $32.29 remaining.

19. What is the % net annual income from money invested in R. R. stocks at 109, which are taxed 1.6% on $\frac{1}{2}$ their par value, if the road pays 4% dividends semi-annually, interest being reckoned on the midyear dividends at 10% for 6 *mo.*?

1 share yields $8.20 annually. Deducting the tax, 80c., leaves the net income on $109, $7.40. Hence the investment yields 6$\frac{86}{109}$%, or nearly 7% net.

20. If I buy 200 N. Y. C. @ 91$\frac{1}{4}$ and sell 200 C. & N. W. short* @ 83$\frac{3}{8}$, and later in the day sell my N. Y. C. @ 94$\frac{5}{8}$ and cover† my C. & N. W. @ 76$\frac{7}{8}$, what is the result of the two operations, brokerage at $\frac{1}{4}$%?

Ans., I made $1875.

The nature of these transactions is that on a certain day I went to the *Stock Exchange* (a place where stocks are bought and sold), and bought 200 shares of the stock of the New York Central Railroad, paying 91\frac{1}{4}$ per share,‡ and later in the day sold the same amount, 200 shares at 94\frac{5}{8}$ per share. The business is really done by a member of the *Stock Board* who is called a *Stock Broker*, and

* This means that I sold what I did not own; *i. e.*, I agreed to furnish so many shares at such a price, and risked my chances of buying them at a lower rate before I should have to deliver them.

† This means that I secured the amount of stock necessary to complete the above bargain; or that I settled with the person to whom I had agreed to furnish this stock.

‡ The money is not paid at the hour of purchase in such transactions, but at the close of the day, if the transaction is a "cash transaction."

when I am said to do anything, he does it for me with my consent. *For all sales which he effects for me I pay him a brokerage of* $\frac{1}{8}$*% on the par value of the stocks.* The other transaction is exactly similar. See foot-notes.

C. & N. W. means Chicago and Northwestern Railroad.

Exchange.

317. A merchant in Detroit wishes to pay a debt of $2500 in New York. He may send the money by a friend, by mail, or by express; but the most common and most convenient way is to step into a *bank* in Detroit, and paying in his $2500 with a small percentage for their trouble, get the Detroit bank's order on a New York bank for the $2500. This *order,* called a *Draft*, the Detroit merchant can send to his creditor in New York, who by stepping into the New York bank to which the order is addressed will get his $2500.

A similar order given by a bank in New York upon a *foreign bank,* as one in London, Eng., would be called a *Bill of Exchange.*

318. Exchange is a method of making payments in distant places by the use of *Drafts* or *Bills of Exchange,* without the direct transmission of money.

When the exchange is between places in the same country, it is called *Inland* or *Domestic Exchange,* and when between places in foreign countries it is called *Foreign Exchange.* Hence a *Draft* is a *Domestic Bill of Exchange.*

319. A ***Draft,*** or ***Bill of Exchange,*** is a written order for money, drawn in one place and payable in another.

320. A ***Bank*** is a company authorized by law to issue paper money, receive deposits, deal in exchange, loan money, or buy and sell coin.

Some banks make it their chief business to loan money, others to deal in exchange, others to receive deposits, while comparatively few are banks of issue, that is, issue paper currency.

DOMESTIC EXCHANGE.

321. [FORMS OF DRAFTS.]

FIRST NATIONAL BANK OF DETROIT,

$2500. DETROIT, MICH., *Feb.* 29, 1876.

At sight, pay to the order of Newcomb, Endicott & Co. TWENTY-FIVE HUNDRED DOLLARS.

SCHUYLER GRANT, *Cashier*.

To the NINTH NATIONAL BANK,
New York, N. Y.

See introductory note to the subject of Exchange. Newcomb, Endicott & Co. are the merchants in Detroit who wish to pay $2500 to their creditor in New York. Having obtained this *draft*, they write on the back of it, "Pay to the order of John Smith," signing this order, "Newcomb, Endicott & Co.," which John Smith is their creditor in New York. When Mr. Smith in New York receives the draft, he takes it to the Ninth National Bank, and writing his own name on the back, receives the money.

$3500. DETROIT, MICH., *Feb.* 29, 1876.

At ten days sight, pay to the order of Sheldon & Co. THIRTY-FIVE HUNDRED DOLLARS, value received, and charge the same to the account of

E. B. SMITH & CO.

To the TWELFTH NATIONAL BANK,
New York.

The first form of draft is of a draft drawn by one bank upon another, the second is a draft drawn by a bookseller's firm upon a bank. Each supposes that the party in Detroit has an account with the bank in New York, and has money in the bank.

Observe that the first above is a *Sight Draft, i. e.*, it is to be paid

as soon as presented to the bank in New York. The second is a *Time Draft*, and is not payable till *ten* days after presentation. It should be presented as soon as received, when the cashier writes on it "Accepted," giving the date of acceptance and signing his name as cashier. This makes the bank liable for it, and is an agreement to pay it after ten days. If no time is specified when a draft is to be paid, it is payable on sight. Drafts are also made payable a certain time after date.

Ex. 1. Suppose New York exchange is at a premium of $\frac{1}{10}\%$ in Detroit, how much will Newcomb, Endicott & Co. have to pay for the above draft (*321*)?

Such exchange is generally at a slight premium in Detroit, since Detroit merchants and other business men want much more New York exchange than New York business men want of Detroit. This requires that Detroit banks should actually send money to the banks in New York, whereas there will be no need of New York banks sending money to Detroit banks. Thus, if the trade between two places is equal, exchange between them should be about at par; but when there is much inequality, exchange on the other place will be at a premium at the place which buys more than it sells, and at a discount at the place which sells more than it buys. Thus there being comparatively little demand in New York for drafts on Detroit, such drafts will be at a slight discount.

In consequence of the above facts, if a New York house wishes to pay a debt in a small Western place, they send to their creditor their check (order) upon a bank in New York, or a Certificate of Deposit. This will be at par in the Western place, and perhaps a little above.

2. What would A. T. Stewart's sight draft on the Twentieth National Bank, N. Y., for \$1500, be worth in Niles, Mich., at $\frac{1}{12}\%$ premium?

3. What will a sight draft for \$1825 on New Orleans sell for in New York, at a discount of $\frac{1}{20}\%$?

4. What will it cost a New York merchant to pay a New Orleans debt of \$2500, New York exchange being at $\frac{1}{10}\%$ premium in New Orleans?

The New York merchant must send a draft which will amount to $2500 including the $\frac{1}{10}\%$, which the New Orleans man will receive as premium. Hence the problem is the ordinary one in percentage, in which the *amount* ($2500), and the rate (.001) are given to find the *base*. Hence we have $b = \frac{A}{1+r} = \frac{\$2500}{1+.001} = \frac{\$2500}{1.001} = \$2497.50+$.

5. A New York merchant sends his creditor in New Orleans a sight draft for $2497.50, New York exchange being at $\frac{1}{10}\%$ premium in New Orleans, what will the New Orleans man receive for the draft?

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6. What is a $1200 St. Louis draft at $^{30}/_{33}$ *da.*, on New York worth, N. Y. exchange being at 101, and the time discount being at 3%?

The nature of this transaction is that a man in St. Louis buys at a bank there a draft on N. Y. for $1200. Since N. Y. exchange is at 101, *i. e.*, at 1% premium, if his draft were a *sight* draft it would cost him $\$1200 \times 1.01 = \$1212$. But inasmuch as the bank in N. Y. will not have to pay the draft till $^{30}/_{33}$ *da.* after its date, they will not charge the St. Louis bank with it till they pay it. Hence the St. Louis bank will have the use of the money $^{30}/_{33}$ *da.* before it will be charged to them in N. Y., *i. e.*, before they have to pay it. Therefore they allow 3% discount on *the face of the draft* for the use of the money. 3% of $1200 for $^{30}/_{33}$ *da.* is $3.25. Hence the purchaser of the draft pays $1212—$3.25, or $1208.75, or $1208.77 including stamp.

7. A merchant in Boston wishes to pay $980 in Milwaukee. Required the cost of a draft, payable in 60 days, exchange being at $1\frac{3}{4}\%$ discount, the bank, or time discount being 4%? *Ans.*, $956.10+.

The nature of this transaction is as follows:

The Boston merchant steps into a bank and buys a draft on a Milwaukee bank. Milwaukee exchange being at discount in Boston, a sight draft for \$980 could be bought for \$980 less $1\frac{3}{4}\%$ of \$980, or for \$962.85. But as the Boston bank will have the money $^{60}/_{63}$ *da.* before it will be charged to them by the Milwaukee bank, and as the Boston merchant will have paid the money $^{60}/_{63}$ *da.* before his creditor in Milwaukee receives it, it is but right that the Boston bank should allow the merchant for the use of the money. Hence they allow him 4% on \$980 for the $^{60}/_{63}$ *da.*, *i. e.*, \$6.766. This deducted from \$962.85 leaves \$956.084, and adding the 2*c.* for the stamp, the cost of the draft is \$956.10+.

The justice of such an arrangement will appear very clearly if we suppose that the debt in Milwaukee is drawing interest. Now no interest will be stopped until $^{60}/_{63}$ *da.* after the debtor has paid his money, hence the party which has the money these 63 days should pay interest on it.

8. Exchange at 2% premium, and time discount at 5%, what is the cost of \$1 exchange on draft for $^{60}/_{63}$ days? Then what of a draft for \$750? For a draft for \$85.50?

9. Exchange at $1\frac{1}{2}\%$ discount, and time discount at 4%, what is the cost of \$1 exchange on draft for $^{90}/_{93}$ *da.*? Then what of a draft for \$500? For \$184.25?

10. Exchange being at $101\frac{3}{4}$ (that is, $1\frac{3}{4}\%$ premium), and time discount at 5%, what % of its face is the cost of a $^{30}/_{33}$ *da.* draft? Of a $^{60}/_{63}$ *da.*? Of a $^{90}/_{93}$ *da.*? Of a 10 *da.*?

11. Exchange being at $98\frac{1}{2}$ ($1\frac{1}{2}\%$ discount), and time discount at $4\frac{1}{2}\%$, what per cent of its face is the cost of a $^{30}/_{33}$ *da.* draft? Of a $^{60}/_{63}$ *da.*? Of a $^{90}/_{93}$ *da.*? Of a 10 *da.*?

322. ***As the rate of exchange at any place, upon the places with which it naturally does business, and the rate for time discount do***

not change frequently, a banker by computing the % of the face for which time drafts can be sold for the usual times, as in Ex. 9, 10, can reserve these as multipliers by which he can readily determine the cost of any draft by a simple multiplication.

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12. A merchant in New York wishes to pay a man in Fort Dodge, Iowa, \$1000, by sending him a sight draft on the Tenth National Bank, New York. New York exchange being at $\frac{1}{5}$% premium in Fort Dodge, for what amount must the draft be drawn?

As the man in Fort Dodge will receive $\frac{1}{5}$% premium on the face of the draft, the draft must be for $\frac{\$1000}{1.002}$.

In such a case as this the New York merchant need not pay in the money at the Tenth National Bank till about the time the draft will come back and be presented, as the draft is supposed to be merely his order on the bank.

13. If in the above case the New York merchant had gone to a broker and bought a draft on the Tenth National for $^{90}/_{93}$ *da.*, at 5%, how much would his draft have cost him?

The face of the draft must be \$998. This he gets at a discount of 5% for $^{90}/_{93}$ *da.*, that is, for \$998—\$12.71 = \$985.29.

14. A merchant in Omaha sold a draft on New York at $\frac{3}{4}$% premium, and received for it \$1259.37$\frac{1}{2}$, what was the face of the draft?

15. To-day, Mar. 5, 1876, New York exchange is quoted in Chicago at 75*c.* discount on \$1000. What does it mean? and at this rate what would be the face of a $^{90}/_{93}$ *da.* draft on New York costing \$2466.28, bank discount being 5%?

16. A shipment of 100 atlases was sent me from Philadelphia, May 1, for which I was to make returns in 90 days. I retailed the maps at \$16 each, and my commission was 40%. At the end of 50 *da.* I find that I have disposed of the atlases, and by remitting the money will be allowed at the rate of $4\frac{1}{2}$% per annum discount. What will a draft cost me at $\frac{1}{8}$% premium?

FOREIGN EXCHANGE.

1. James Howell, a young man from Chicago, is traveling in England, and his father, Thomas Howell, wishes to send to him in London \$1000. How will he effect it? and what amount in English currency will the son receive, sterling exchange being quoted at $4.89\frac{1}{2}$, and gold at 114?

Answer.—The father may go to a Chicago bank which deals in foreign exchange, and get a *bill of exchange* on London. (This is practically the same thing as a draft. See introductory note.) The meaning of the quotation is that he will have to pay $\$4.89\frac{1}{2}$ in gold for every *pound*, face of the bill But as gold is at 114, \$1000 in currency is equivalent to $\frac{\$1000}{114} = \$877.19$ in gold. Hence the face of the bill will be $\frac{877.19}{4.895} = 179.2+$, or £179 4*s.*

323. Mr. Howell will receive from the bank *three* bills of exchange (orders on the London bank) of the following form:

£179 4*s.* CHICAGO, ILL., *March* 7, 1876.

At sight of this FIRST of Exchange (Second and Third of same date and tenor unpaid), pay to the order of James Howell, *One hundred and seventy-nine pounds and four shillings sterling,* value received, and charge the same to

BROWN, GALE & CO.

To SUNDERLAND & HATCH, *London.*

The other two bills will be exactly like this, except that in the second the word SECOND will be used where FIRST is in this, and the parenthesis will read, "First and Third of same date," etc. The third will read, "THIRD of exchange (First and second of same date, etc.)."

The object of this arrangement is that the three bills may be sent by different mails, and thus if one is lost the remittance will not fail. Of course when one has been received and paid, the others are void.

324. *Quotations* are the statements made from day to day in the *newspapers,* giving the rates at which exchange, stocks, bonds, etc., are being bought and sold in the money market. These quotations are *gold values.*

Quotations of London exchange give the value of £1 in dollars. The par value of £1 being $4.8665, when London exchange is quoted more than this it is at *premium;* when less, at *discount.*

Quotations of Paris, Antwerp, and Geneva exchange give the value of $1 in *francs.* The par value of a franc being .193, $1 = 5.18 *francs* +. Hence when quotations are less than 5.18 this exchange is at premium; when greater, at discount.

Quotations of Hamburg, Frankfort, Bremen, and Berlin exchange give the value of 4 *marks* in *cents.* The par value of 4 *marks* being 95.2 *cents,* when the quotation is 97, is Berlin exchange at premium, or discount? When it is 94?

Quotations of Amsterdam exchange give the value of a *guilder* (par 39 cents) in cents. When do the quotations show Amsterdam exchange to be at premium? When at discount?

2. A Detroit young man is in Berlin, when there is paid in at David Preston & Co.'s bank in Detroit $2500 in currency on his account, which he has ordered remitted to him by bill of exchange on Berlin. Berlin exchange being quoted in the papers at 93½, and gold 115½, and the banker's brokerage being ⅛%, what amount will the young man in Berlin receive?

First reduce the \$2500 currency to gold. \$2500 *currency* = $\frac{\$2500}{1.155}$ = \$2164.50 *gold*. As 4 *marks* exchange can be bought for $93\frac{1}{2}$ *cents* gold (exclusive of brokerage), \$2164.50 *gold* = $\frac{2164.50 \times 4}{.935}$ = 9259.89 *marks*. But Preston & Co.'s brokerage being $\frac{1}{8}$% *on the face of the bill* (***314***), it requires 1 *mark* + $\frac{1}{8}$% of 1 *mark* for every mark remitted, or $1.00\frac{1}{8}$ *marks*. Therefore the face of the bill will be $\frac{9259.89}{1.00\frac{1}{8}}$ = 9248.32 *marks*, which is what the young man receives.

3. Quotations being, gold 114, and Paris exchange 5.15, and brokerage being $\frac{1}{10}$%, what will be the cost in currency of a bill on Paris for 10000 *fr.*?

10000 *francs* = $\frac{10000}{5.15}$ = \$1941.75 in *gold*. \$1941.75 in gold = \$1941.75 × 1.14 = \$2213.595 in currency. This is the face of the bill in *currency*. On this the brokerage of $\frac{1}{8}$% is \$2.77. Hence the cost of the bill in currency is \$2213.59 + \$2.77, or \$2216.36.

4. Gold being 110, and Amsterdam exchange $40\frac{3}{4}$, and brokerage $\frac{1}{8}$%, what cost a bill for 5000 guilders?

5. With gold at 112, and sterling exchange at 4.88, and brokerage $\frac{1}{4}$%, how large a bill on London can be obtained for \$5000?

6. When Geneva exchange is quoted at 5.14, what is the per cent premium? What when Amsterdam is quoted $40\frac{3}{4}$? What when sterling is quoted 4.88?

7. When London exchange is 2% premium, what is the quotation? When Paris exchange is $1\frac{3}{4}$% premium, what is the quotation? What is the quotation when Berlin exchange is at $2\frac{1}{2}$% premium?

325. The price at which exchange is quoted is often called ***The Course of Exchange.***

8. When a bill on Paris for 495 francs costs $96.12 in gold (exclusive of brokerage), what is the course of exchange? *i. e.*, How many francs exchange can be bought for $1? *Ans.*, 5.15.

9. If a bill on Hamburg for 2155 marks costs $566.50 in currency (exclusive of brokerage), gold being at 110¼, what is the quotation of Hamburg exchange?

Ans., 95⅝.

10. If a bill on Frankfort for 13550 marks costs $3500.40 in currency, the course of exchange being 94¼, and brokerage ⅛%, what is the quotation of gold? *Ans.*, 109¼.

11. A dealer in foreign exchange said: "When Amsterdam is quoted at 40, exchange is 2% premium." What did he reckon the gold value of a guilder?

12. In the fall of 1870 a patriotic Frenchman in New York wishing to aid his native land, proposed to invest $5000 U. S. currency in the 6% French loan, then being sold in London at 85. Gold being 110, London exchange 4.87, brokerage in New York ¼%, commission for buying in London ½%, and London exchange on Paris being 25.43, what % will the Frenchman receive on his investment for 1875, remembering that he will receive the rentes (interest) in gold in France, and reckoning exchange on Paris at 5.15, and gold at 115?

The $5000 will secure a bill on London for £931.03. The commission for buying being ½% leaves £926.40 to be invested. The London exchange on Paris being 25.43, *i. e.*, a pound being worth 25.43 francs, £926.40 = 23558.35 francs. Now 23558.35 francs will secure $\frac{100}{85}$ of 23558.35 francs, or 27715.7 francs *nominal capital* in the loan. On this the Frenchman receives for 1875, 1662.94 francs rentes (interest), which in currency in New York is worth $\frac{1662.94 \times 1.15}{5.15}$ = $371.34, or a little over 7⅜% on his investment.

Equation of Payments.

326. ***Equation of Payments*** is the process of finding the mean or average maturity or date of several obligations.

327. The ***Term of Credit*** is the period from the date to the maturity of an obligation.

328. The ***Equated Time*** is the equitable date for the payment of several obligations maturing at different dates.

329. The following rule covers all cases of Equation of Payments and Averaging Accounts:

RULE.—*Find the interest which would accrue on each obligation (at any rate per cent., when none is named), from its maturity to the most remote maturity. Then ascertain how long it would take the sum of the obligations, or the unpaid balance, to produce the sum of these interests, or the balance of interest, at the same rate per cent. Subtract this time from the date of the most remote maturity, or add it as the case may require, and the result will give the Equated Time.*

Ex. 1. On the 1st day of January, 1870, I buy of Mr. B a piece of land, for which I agree to pay him $200 in 2 *yr.*, $300 in 4 *yr.*, and $400 in 6 *yr.*, *without interest.* Soon after closing the contract I find that it will be more convenient for me to pay the whole $900 at one time. B agreeing to this, at what date should the payment be made that neither party gain or lose by the change of contract?

Had I paid it as originally agreed, money being worth 10%, Mr. B would have had $80 interest on the $200 payment, and $60 on

the $300 payment at the expiration of the 6 *yr.* Hence the question is, How long before the expiration of the 6 *yr.* must I pay the $900 in order that it may accumulate $140 interest at 10% at the expiration of that time? As $900 yields $90 interest in 1 *yr.* at 10%, to yield $140 requires $\frac{140}{90}$, or $1\frac{5}{9}$ *yr.* Hence the *Equated Time* is $1\frac{5}{9}$ *yr. before* Jan. 1, 1876, *i. e.*, June 11, 1874.

(*a*) From this solution we readily perceive that the rate of interest we assume is immaterial, since, if we take twice as great a rate, twice as much interest will be estimated on the payments, and twice as much on the sum of the payments for a unit of time.

(*b*) By means of Interest Tables and of a table like the following, the operations in Equation of payments become very simple; especially is this the case if the Interest Tables are so arranged that the time required for any sum to yield any given interest can be taken directly from them.*

330. *Table giving the number of Days between any day of any month and the corresponding day of any other month in a year.*

	Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec
Jan.	365	31	59	90	120	151	181	212	243	273	304	334
Feb.	334	365	28	59	89	120	150	181	212	242	273	303
Mar.	306	337	365	31	61	92	122	153	184	214	245	275
Apr.	275	306	334	365	30	61	91	122	153	183	214	244
May.	245	276	304	335	365	31	61	92	123	153	184	214
June.	214	245	273	304	334	365	30	61	92	122	153	183
July.	184	215	243	274	304	335	365	31	62	92	123	153
Aug.	153	184	212	243	273	304	334	365	31	61	92	122
Sept.	122	153	181	212	242	273	303	334	365	30	61	91
Oct.	92	123	151	182	212	243	273	304	335	365	31	61
Nov.	61	92	120	151	181	212	242	273	304	334	365	30
Dec.	31	62	90	121	151	182	212	243	274	304	335	365

* A set of Interest Tables having this very desirable feature is now being prepared by Dr. Watson, Director of the Observatory of the University of Michigan. These tables will also conform to the ruling of our courts and to the statutes making a day $\frac{1}{365}$ part of a year.

NOTE.—To find how many days between May 17 and Sept. 25: The table shows that from May 17 to Sept. 17 is 123 *da.*; hence, to Sept. 25 is 123 + 8 = 131 *da.*

To find how many days from July 27 to Dec. 10: The table shows that from July 27 to Dec. 27 is 153 *da.*; hence, to Dec. 10 is 153 − 17 = 136 *da.*

To find a date 208 *da. earlier* than Nov. 20: The table shows that from Apr. 20 to Nov. 20 is 214 *da.*; hence, 6 *da.* less interval gives Apr. 26.

To find a date 156 *da.* later than Oct. 28: The table shows that from Oct. 28 to Mar. 28 is 151 *da.*; hence, 5 *da.* more gives Apr. 2.

2. May 10, 1875, Mr. A buys a horse of Mr. B, agreeing to pay $30 in 2 *mo.*, $50 in 3 *mo.*, $40 in 5 *mo.*, and $60 in 6 *mo.*, without interest. Before the first payment falls due, he agrees with B to pay the $180 at one time. What should be the date of payment in order that neither party gain or lose by the change of contract? *Ans.* Sept. 18, 1875.

$30 matures July 10.	To Nov. 10 is 123 *da.*	Int., 10%,* is $1.01
$50 matures Aug. 10.	To Nov. 10 is 92 *da.*	Int., 10%, is 1.26
$40 matures Oct. 10.	To Nov. 10 is 31 *da.*	Int., 10%, is .34
$60 matures Nov. 10.		$2.61
$180. Interest 1 *da.* at 10%, 4.93 cents.		

261 ÷ 4.93 = 53 −. Hence, the equated time is 53 *da.* antecedent to Nov. 10. From the table we see that this is Sept. 18.

3. What would be the equitable time for the payment of the several sums in the last example if all bore interest from May 10, 1875, at 10%, which was all money was worth?

4. What would be the equitable date for the payment of the *face* of the above obligations ($180) if they were on interest at 6%, and money was worth 10%?

* When Interest Tables are not at hand, 10% is the most convenient rate to reckon, except when the time is in *months*; then 6%, being ½ a cent a month, is most convenient.

Being on interest at 6%, they would yield at maturity, if paid as agreed, \$30.30, \$50.75, \$41, and \$61.80. Now, as money is worth 10%, B should be allowed 10% on these sums to the end of the 6 *mo.* This gives \$31.32 + \$52.03 + \$41.35 + \$61.80 = \$186.50 as the amount which B would receive Nov. 10, 1875, according to the conditions; *i. e.*, \$6.50 interest would accrue. Hence the question is, How long will it take \$180 at 10% to yield \$6.50 interest? This is found to be 132 *da.* Hence the equated time is July 1, 1875.

5. A Fort Dodge, Iowa, merchant having made 3 bills with a New York merchant on May 5, 1876, one for \$300, on which he is to have 60 *da.* credit, one for \$500, on which he is to have 120 *da.*, and one for \$750, on which he is to have 180 *da.*, all without interest, he gives his note for the entire sum, "on interest at 6% *after due.*" When should the note be made payable?

Ans., Sept. $^{19}/_{22}$, 1876.

6. April 10, 1875, bought a bill of goods amounting to \$1500 on 6 *mo.* credit. June 10, 1875, paid \$300, and Aug. 10, 1875, \$400. When was the balance due?

Paying the \$300 4 *mo.* before due, I am entitled to interest on it (say at 10%, since the rate is immaterial, see (*a*) under Ex. 1). This is \$10. In like manner I am entitled to \$6.67 interest on the \$400 paid 2 *mo.* before due. Hence the question is, How long can I retain the unpaid balance, \$800? *i. e.*, how long will it take \$800 at 10% to yield \$16.67 interest? This is 76 *da.* Hence the equated time is 76 *da.* after the original maturity. 6 *mo.* after Apr. 10 is Oct. 10; and from the table we learn that from Oct. 10 to Dec. 10 is 61 *da.*, and 15 *da.* more brings the date to Dec. 25.

7. May 6, 1874, I bought 4 bills mdse. as follows: \$380 cash; \$420 on 3 *mo.*; \$560 on 4 *mo.*, and \$600 on 5 *mo.* What was the equated time? *Ans.*, Aug. 16, 1874.

The \$380, for which I was to pay cash down, being put in with the others and paid with them, will be reckoned as \$380 on 0 *da. credit.*

Interest on	$380 for 5 *mo.*, at 6%,*	$9.50
" "	$420 for 2 *mo.*, at 6%, ...	$4.20
" "	$560 for 1 *mo.*, at 6%	$2.80
	$600	$16.50

Interest on $1960, 1 *mo.*, $9.80. Hence the equated time is $\frac{16.5}{9.8}$ = 1.6838+, or 1 *mo.* 21 *da.* before Oct. 6, *i. e.*, Aug. 16.

8. Bought of Sheldon & Co. the following bills of mdse:

1875. Jan. 2, To mdse. on 1 *mo.* $420.
Feb. 20, " " 3 *mo.* $420.
Mar. 15, " " 6 *mo.* $910.

What is the equated time of payment?

Ans., June 25.

Find the date of the maturity of each obligation, when this problem becomes the same as the preceding. It is thus equivalent to the following: Bought $420 at cash, $420 at 107 *da.*, and $910 at 225 *da.*

9. Bought goods of A. T. Stewart & Co. as follows:

1864. May 5, To mdse. on 2 *mo.* $860.
June 20, " " 4 *mo.* $480.
July 10, " " 5 *mo.* $760.

What is the equated time of payment of these bills?

Ans., Sept. 25.

10. A owes $600, due in 6 months: 4 months before it is due $200 is paid, and 2 months before it is due $200 more is paid. How long after the expiration of the 6 months may the remaining $200 remain unpaid?

* When Interest Tables are not at hand and the time is in months, 6%, being ½ a cent a month, is most readily reckoned; and in fact, when the time is in months, this is even more expeditious than to use the tables. When the time is in days, it may be thought that 6% is also equally convenient, since ⅙ of the days reckoned as mills is often taken as the interest on $1. But this is inaccurate, as it is based on a day as $\frac{1}{360}$ of a year.

331. AVERAGING ACCOUNTS.

1. *Dr.* ELIAS HOWE. *Cr.*

1875.				1875.			
June 5.	Mdse.	$400	00	July 1.	Cash.	$350	00
Aug. 12.	" 3 *mo.*	$600	00	Aug. 25.	Draft at 60 *da.*	$500	00
Sept. 8.	" 6 *mo.*	$500	00	Sept. 10.	Cash.	$600	00
Sept. 20.	" 4 *mo.*	$1000	00	Sept. 15.	Draft at 30 *da.*	$250	00

What is the equated time for the payment of the balance of this account?

The *Debits* in this account fall due as follows:

June 5, 1875, $400, 276 *da.*, ... before Mar. 8.
Nov. 12, " $600, 116 *da.*, ... " "
Jan. 20, 1876, $1000, 47 *da.*, ... " "
Mar. 8, " $500, 0 *da.*, hence due Mar. 8.

The interest which would accrue on these up to Mar. 8, 1876, at 10%, is $62.19.

The receipts on the account were:

July 1, 1875, $350, 250 *da.*, ... before Mar. 8.
Sept. 10, " $600, 179 *da.*, ... " "
Oct. 18, " $250, 141 *da.*, ... " "
Oct. 27, " $500, 132 *da.*, ... " "

The interest on these sums up to Mar. 8, 1876, at 10%, is $81.135, giving $81.135 – $62.19 = $18.945, in favor of the debtor, Mr. Howe. Hence the question becomes, How long can the balance of the account, $800, be retained by the debtor in order to offset this interest, *i. e.*, how long will it take $800 at 10% to yield $18.945? This is 86 *da.*; and the equated time is June 2, 1876.

2. Required the balance due on the following account, *Dec.* 1, 1874, at 6%:

Dr.		ASA SMITH.				*Cr.*	
1874.				1874.			
Mar. 14.	To mdse. 6 *mo.*	$560	00	June 10.	By mdse. 2 *mo.*	$600	00
April 20.	" 6 *mo.*	$650	00	Aug. 5.	By Cash.	$400	00
May 10.	To Cash.	$540	00	Sept. 20.	By mdse. 1 *mo.*	$300	00
June 15.	To Cash.	$350	00	Nov. 20.	By Cash.	$300	00

Ans. Asa Smith owes his creditor $500, the equated time for the payment of which is April 26, 1874. Hence, on Dec. 1, 1874, Mr. Smith owes on this account the amount of $500 for 219 *da.*, at 6 %, or $518.

As facility in solving such problems depends much upon the *form of solution*, we give the following

ARRANGEMENT OF OPERATION.

Maturity of Debits.		*Days before Latest Maturity.*	*Debit Int.*	*Maturity of Credits.*		*Days before Latest Maturity.*	*Credit Int.*
May 10,	$540, . .	194, . .	$28.70	Aug. 5,	$400, . .	107, . .	$11.73
June 15,	350, . .	158, . .	15.15	Aug. 10,	600, . .	102, . .	16.77
Sept. 14,	560, . .	67, . .	10.28	Oct. 20,	300, . .	31, . .	2.55
Oct. 20,	650, . .	31, . .	5.52	Nov. 20,	300		
	$2100		$59.65		$1600		$31.05
	1600		31.05				
	$500		$28.60				

The maturities will be written directly from the account, and the times and interests taken from tables, if these are at hand. If then we have Interest Tables so arranged that we can find directly from them the time it takes $500 at 10% (which is used in this solution) to yield $28.60, we are enabled to find the equated time of such an account with very little computation, and this only requiring simple addition.

3. What is the cash balance of the following account, Sept. 20th, 1874, at 6%?

Dr.	Bach & Abel.				*Cr.*
1874.			1874.		
April 10.	To mdse. 2 *mo.*	$800	April 20.	By Cash.	$400
April 30.	To Cash.	$600	May 10.	By mdse. 2 *mo.*	$600
May 15.	To mdse. 2 *mo.*	$700	June 30	By mdse. 2 *mo.*	$500
June 5.	To mdse.	$200	Aug. 10	By Cash.	$500

4. What was due on the following account Jan. 1, 1873, interest being 5%, and 4 *mo.* credit being allowed on each entry?

Dr.	David H. Daniels in % with Geo. W. Dean.						*Cr.*
1872.				1872.			
July 8.	To mdse.	$236	17	July 3.	By mdse.	$439	27
Aug. 1.	To sundries.	$819	63	July 25.	By mdse.	$213	16
Sept. 4.	To mdse.	$142	13	Sept. 13.	By mdse.	$100	00
Nov. 13.	To mdse.	$947	22	Oct. 24.	By mdse.	$262	18
Dec. 8.	To sundries.	$1050	00	Nov. 30.	By mdse.	$327	48
				Dec. 21.	By mdse.	$520	75

5. White & Co. of Rochester, N. Y., buyers of wheat, forwarded to Hall & Jones, commission merchants in New York city—

1875.

Sept. 20, 2000 *bu.* wheat, sold by H. & J. at $1.75
Oct. 1, 2500 *bu.* " " " " $1.80
Nov. 15, 3000 *bu.* " " " " $1.70

On the above Hall & Jones paid 5*c.* per bushel freight 3 *da.* after the shipment of each lot. They sold the first lot 2 *mo.* after it was sent, the second $3\frac{1}{2}$ *mo.*, and the third 40 *da.*, paying 2*c.* per bushel storage for the first 20 *da.*, or part of the same, and $\frac{1}{2}$*c.* for each additional 10 *da.*, or part thereof, after 20 *da.*; 1*c.* per bushel insurance paid when the grain was admitted to store, and $1\frac{3}{4}$*c.* per bushel

commission for selling. What was the cash balance due White & Co. Mar. 1, Hall & Jones having sent them a certificate of deposit of \$3000 Jan. 1, which was worth $\frac{1}{10}$% premium in Rochester, allowing 7% for use of money between the equated time and Mar. 1?

Property Insurance.

332. *Insurance* is a branch of business in which companies called *Insurance Companies* make contracts to pay specified sums of money to other parties, in the event of certain losses to which the latter may be liable, the company receiving a percentage on the sum guaranteed.

The contract is called a ***Policy.*** The sum which the party insured pays to the company is called the ***Premium.***

The company or party which insures is sometimes called the *Underwriter.*

333. There are two principal departments of the Insurance Business; viz., *Property Insurance* and *Life Insurance.*

[For *Life Insurance* see next section.]

The principal department of Property Insurance is *Fire Insurance,* in which the guarantee is against loss by fire. *Marine Insurance* is insurance against the contingencies of loss to vessels or property on them.

By long and careful observation it is known that, though no one can tell whether a particular piece of property, as a house, a barn, or a ship, may not be destroyed by fire the next hour, we can tell about how much property will be destroyed by fire in a given region of country in any year. It is upon such knowledge that *Property Insurance* is based. Thus, if it is probable that not more than $\frac{1}{10}$% of the property in any given region will be destroyed by fire in a year, I can safely make contracts with *all* the individuals in that region that if they will pay me $\frac{1}{5}$% annually on $\frac{2}{3}$ of their prop-

erty, I will refund to them $\frac{3}{4}$ the value of their property in the event of its being destroyed by fire. Of course it is possible that a long series of years may elapse and little or no property be destroyed by fire in that region, and it is also possible that a large part, or even all the property in that region may be swept away by fire in a few days. In the former case I would accumulate money by the operation, and in the latter be ruined.

334. Insurance companies are of two principal kinds, *Stock* and *Mutual.* In the former the profits are shared and the losses borne by the stockholders in the ratio of their stock, after paying the expenses of the company; in the latter these are shared by the *Insured* or *Policy Holders,* who are in fact the stockholders.

Property is not usually insured for more than $\frac{3}{4}$ its market value, and specially endangered property will not be insured at all by judicious companies. Before the Policy is given the property has to be carefully surveyed and described with respect to the dangers to which it is exposed, and any deception or fraud in this respect, on the part of the owner, vitiates the policy.

Ex. 1. My house is valued at $5000, my furniture at $2000, and my library at $2000. If I get the whole insured for $\frac{2}{3}$ its value at $\frac{3}{8}\%$, what is my annual premium? In the event of damage by fire, if my house is injured to the extent of $1500, my furniture $600, and my library $300, what shall I receive from the company?

Ans. Premium, $22.50. Ordinarily, policies make the company liable to pay all damages up to the amount insured. Hence in this case I should receive $2400.

Companies often reserve the right to restore the damaged building in lieu of paying its estimated damage in money to the owner.

2. I insure $1700 on my house, $200 on my furniture, and $100 on my library, for 3 years, paying $18. What is the rate per annum?

3. The premium for insuring a tannery for $\frac{3}{4}$ of its value, at $1\frac{2}{3}\%$, was $145.60; required the value of the tannery.

4. At $1\frac{3}{8}\%$, the premium for insuring my store was \$89.10 ; required the amount of the insurance.

5. A Milwaukee grain dealer insured a cargo of 3000 bushels of wheat, valued at \$1.50 per bushel, and bound for Oswego, N. Y., at $3\frac{1}{2}\%$, the policy covering both the value of the wheat and the premium. What was the amount of the policy?

6. At what % must I insure \$8000 worth of property for $\frac{3}{4}$ its value, so that the premium for 3 years shall be \$50, including \$1 for policy and \$1 for the survey ?

7. Jan. 1, 1870, I took out a policy on my house, valued at \$6000, covering $\frac{3}{4}$ its worth, at $\frac{3}{5}\%$, and paid \$1 for policy and \$1 for surveying. I carried the policy 6 years and 3 *mo.*, when the house burned. What was my loss, reckoning 7% interest on the annual premiums?

8. If it cost $\$37\frac{1}{2}$ to insure my barn and grain, at $1\frac{1}{2}\%$, for $\frac{2}{3}$ their worth, what was their value?

9. A cargo of flour, valued at \$6.25 per barrel, was insured in transitu from New York to Liverpool at $12\frac{1}{2}$*c.* per barrel. What per cent. was it?

10. A policy covering value of property and premium was taken at $1\frac{1}{2}\%$; premium, \$17.50. What was the value of the property?

SECTION IV.

ANNUITIES AND LIFE INSURANCE.

335. An ***Annuity*** is a sum of money payable annually, or at other regular intervals.

Pensions, Dowers, Rents, &c., are examples of annuities.

336. A *Certain Annuity* is one which continues for a limited time. A *Perpetual Annuity,* or a *Perpetuity,* continues forever. A *Contingent Annuity* is one whose commencement, or duration, or both, is limited, as by a person's death, or by his arrival at a certain age.

An Annuity in *Reversion,* or *Deferred,* is one which begins at some future time. An Annuity in *Arrears,* or *Forborne,* is one the payment of which has not been made when due.

337. ***The Amount*** of an annuity is the sum of all the payments, plus the interest of each payment, from the time it became due.

338. ***The Present Worth*** of an annuity is such a sum of money as will, in the given time and rate %, amount to the final value.

Certain Annuities.

Ex. 1. What is the present worth of an *Annuity Certain* of $120 for 4 years, money being worth *7% simple interest?* What, if money is reckoned at *7% compound interest?*

As the Annuitant receives $120 at the close of each year for 4 years, we have the *amount* of the annuity at simple interest equal to $120 + int. of $120 for 3 *yr.* + $120 + int. of $120 for 2 *yr.*+ $120+int. of $120 for 1 *yr.*+$120, or $530.40.

Now the question is, What is the present worth of $530.40 due 4 *yr.* hence, money being worth 7%, simple interest? This is $414.375.

This, however, is a view rarely or never taken of annuities. Computations concerning them are always made on the basis of *compound interest.* Thus we have in this case the amount of $120 at compound interest (7%) for 3 *yr.*+the amount of $120 at comp. int. for 2 *yr.* + the amt. for 1 *yr.* + $120, or $532.79316. This amount is therefore to be discounted at compound interest for 4 *yr.*, *at 7%,* giving as the *Present Worth,* $406.465+.

2. To produce the *formulæ* for the *Amount*, and the *Present Worth* of an *Annuity Certain*, at *Simple Interest:*

SOLUTION.—Let a be the *annuity* (the sum due at the end of each year), r the *rate*, t the *time*, A the *Amount*, and P_w the *Present Worth*.

1st. *To find the Amount.*—The *last* payment not being on interest is a. The next to the last, on interest 1 *yr.*, amounts to $a(1 + r)$; the next preceding, on interest 2 *yr.*, amounts to $a(1 + 2r)$; the next preceding to $a(1 + 3r)$, etc., to the first payment, which being on interest $t - 1$ years, amounts to $a[1 + (t - 1)r]$. Hence we have,

$$A = a\{1 + [1+r] + [1+2r] + [1+3r] +, \text{etc., to } [1+(t-1)r]\}$$

Now the factor in the brackets is the sum of an arithmetical progression, the first term of which is 1, the common difference r, and the number of terms t. This sum is $\frac{t}{2}[2+(t-1)r]$. Hence the formula required is,

$$(1) \qquad A = \frac{t}{2}[2+(t-1)r]a.$$

2d. *To find the Present Worth.*—Since the present worth is the amount divided by the amount of 1 for the given rate and time, we have,

$$(2) \qquad P_w = \frac{\frac{t}{2}[2+(t-1)r]}{1+rt}a.$$

3. What is the present worth of a pension of \$800 a year for 8 *yr.* at 7% simple interest? What the amount?

By the formula, $P_w = \frac{4(2+.49)}{1.56}800 = \$5107.69.$

Also analyze as under Ex. 1.

4. Which is the better, an annual rent of \$500 for 10 *yr.*, or a cash payment in advance of \$3500, money being worth 6% simple interest?

Ans., The former, by \$468.75.

5. To produce the *formulæ* for the *Amount,* and the *Present Worth* of an *Annuity Certain,* at *Compound Interest:*

SOLUTION.—1st. *To find the Amount.*—With the same notation as above, and with a similar line of thought, substituting compound for simple interest, we have,

Last payment, ...	a.
Amount of next to the last, ...	$a(1+r)$.
" " second from last, ...	$a(1+r)^2$.
" " third " "	$a(1+r)^3$.
etc., etc., to the first.	
Amount of first payment, ...	$a(1+r)^{t-1}$.

Hence $A = a[1+(1+r)+(1+r)^2+(1+r)^3+$, etc., to $(1+r)^{t-1}]$. Now the factor within the brackets is the sum of a *geometrical progression,* the first term of which is 1, the ratio r, the number of terms t, and hence the last term $(1+r)^{t-1}$. The sum of this series is $\frac{(1+r)^t-1}{r}$. Hence the formula required is,

(1) $$A = \frac{(1+r)^t-1}{r}a.$$

2d. *To find the Present Worth.*—For this we are to divide the amount by the amount of 1 for the given rate and time at compound interest, *i. e.*, by $(1+r)^t$. Hence $P_w = \frac{(1+r)^t-1}{r(1+r)^t}a$.

Observe that this formula is identical with $x = \frac{br(1+r)^n}{(1-r)^n-1}$, of (*281*), which, solved for b, gives $b = \frac{(1+r)^n-1}{r(1+r)^n}x$. Why is this? What propriety is there in calling the former a *Simple Interest* formula, and this a *Compound Interest* formula?

6. What is the present worth of an Annuity Certain of \$800 for 8 *yr.*, at 7%, at compound interest? What the amount?

By the formula, $P_w = \frac{(1+r)^t-1}{r(1+r)^t}a = \frac{(1.07)^8-1}{.07(1.07)^8}800 = \$4777.04-$.
Also analyze as under Ex. 1.

Compare Ex's 3 and 6. Why does compound interest give a less present worth? Which gives the greater *amount*?

7. What sum in hand is equivalent to an annuity of \$500 a year for 10 *yr.*, money being worth 10% compound interest?

8. At the birth of a son a father invests for him \$100 at 5% annual interest, and the same amount with accrued interest on each birthday thereafter till the son is 21 *yr.* old. What sum will the son have when he comes of age?

TABLES.

Observe that $A = \frac{(1+r)^t - 1}{r} a$, becomes $A = \frac{(1+r)^t - 1}{r}$, for $a = 1$.

For $a = 200$ we have $A = \frac{(1+r)^t - 1}{r} 200$, etc. Hence if we have a *table* giving the amount of \$1 annuity for given rates and times, we can readily find from it the amount of any annuity for corresponding rates and times by multiplying the amount of \$1 by the given annuity.

339. *Table giving the Amount of \$1 Annuity.*

Yrs.	3%.	3½%.	4%.	5%.	6%.	7%.	8%.	10%.
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	2.0300	2.0350	2.0400	2.0500	2.0600	2.0700	2.0800	2.1000
3	3.0909	3.1062	3.1216	3.1525	3.1836	3.2149	3.2464	3.3100
4	4.1836	4.2149	4.2465	4.3101	4.3746	4.4399	4.5061	4.6410
5	5.3091	5.3625	5.4163	5.5256	5.6371	5.7507	5.8666	6.1051
6	6.4684	6.5502	6.6330	6.8019	6.9753	7.1533	7.3359	7.7156
7	7.6625	7.7794	7.8983	8.1420	8.3938	8.6540	8.9228	9.4872
8	8.8923	9.0517	9.2142	9.5491	9.8975	10.2598	10.6366	11.4359
9	10.1591	10.3685	10.5828	11.0266	11.4913	11.9780	12.4876	13.5795
10	11.4639	11.7314	12.0061	12.5779	13.1808	13.8164	14.4866	15.9374
11	12.8078	13.1420	13.4864	14.2068	14.9716	15.7836	16.6455	18.5312
12	14.1920	14.6020	15.0258	15.9171	16.8699	17.8885	18.9771	21.3843
13	15.6178	16.1130	16.6268	17.7130	18.8821	20.1406	21.4953	24.5227
14	17.0863	17.6770	18.2919	19.5986	21.0151	22.5505	24.2149	27.9750
15	18.5989	19.2957	20.0236	21.5786	23.2760	25.1290	27.1521	31.7725
16	20.1569	20.9710	21.8245	23,6575	25.6725	27.8881	30.3243	35.9497
17	21.7616	22.7050	23.6975	25.8404	28.2129	30.8402	33.7502	40.5447
18	23.4144	24.4997	25.6454	28.1324	30.9057	33.9990	37.4502	45.5992
19	25.1169	26.3572	27.6712	30.5390	33.7600	37.3790	41.4463	51.1591
20	26.8704	28.2797	29.7781	33.0660	36.7856	40.9955	45.7620	57.2750

In like manner, from $P_w = \dfrac{(1+r)^t-1}{r(1+r)^t}a$, we see that the factor $\dfrac{(1+r)^t-1}{r(1+r)^t}$ is the present worth of $1 annuity for time t at rate r, and hence that, if we have a table giving the present worth of $1 annuity for given rates and times, we can find the present worth of any annuity by multiplying the corresponding present worth of $1 by the annuity.

340. Table giving the Present Worth of $1 Annuity.

Yrs.	3%.	3½%.	4%.	5%.	6%.	7%.	8%.	10%.
1	0.9709	0.9662	0.9615	0.9524	0.9434	0.9346	0.9259	0.9091
2	1.9135	1.8997	1.8861	1.8594	1.8334	1.8080	1.7833	1.7355
3	2.8286	2.8016	2.7751	2.7232	2.6730	2.6243	2.5771	2.4869
4	3.7171	3.6731	3.6299	3.5460	3.4651	3.3872	3.3121	3.1699
5	4.5797	4.5151	4.4518	4.3295	4.2124	4.1002	3.9927	3.7908
6	5.4172	5.3286	5.2421	5.0757	4.9173	4.7665	4.6229	4.3553
7	6.2303	6.1145	6.0021	5.7864	5.5824	5.3893	5.2064	4.8684
8	7.0197	6.8740	6.7327	6.4632	6.2098	5.9713	5.7466	5.3349
9	7.7861	7.6077	7.4353	7.1078	6.8017	6.5152	6.2469	5.7590
10	8.5302	8.3166	8.1109	7.7217	7.3601	7.0236	6.7101	6.1446
11	9.2526	9,0016	8.7605	8.3064	7.8869	7.4987	7.1390	6.4951
12	9.9540	9.6633	9.3851	8.8633	8.3838	7.9427	7.5361	6.8137
13	10.6350	10.3027	9.9856	9.3936	8.8527	8.3577	7.9038	7.1034
14	11.2961	10.9205	10.5631	9.8986	9.2950	8.7455	8.2442	7.3667
15	11.9379	11.5174	11.1184	10.3797	9.7122	9.1079	8.5595	7.6061
16	12.5611	12.0941	11.6523	10.8378	10.1059	9.4466	8.8514	7.8237
17	13.1661	12.6513	12.1657	11.2741	10.4773	9.7632	9.1216	8.0216
18	13.7535	13.1897	12.6593	11.6896	10.8276	10.0591	9.3719	8.2014
19	14.3238	13.7098	13.1339	12.0853	11,1581	10.3356	9.6036	8.3649
20	14.8775	14.2124	13.5903	12.4622	11.4699	10.5940	9.8181	8.5136

QUERIES—1. Why is it that the first of these tables can be made from the Compound Interest Table (*269*) as follows: looking at the 3% columns in each, we observe that the number opposite 2 in the table above is that opposite 1 in the other table + 1; the number opposite 3 in this table is the sum of the preceding in (*269*) + 1; and so *the number opposite any figure in this is the sum of the preceding* in (*269*) + 1?

2. Why is it that, dividing any amount taken from (*339*) by the corresponding amount from (*269*), gives the corresponding result in (*340*)? Show that these facts appear in the *formulæ*.

9. Find from the above tables the amount of $500 annuity for 15 *yr.*, at 8%; at 3%; at 6%. Find also the present worth for 6 *yr.*; for 9 *yr.*; for 20 *yr.*, at 5%.

10. A man at 60 finds himself unable to work, and in destitution; but since he was 15 years old he has used on an average 20 cents worth of tobacco per day. Instead of using the tobacco, suppose he had saved the money and invested it at the close of each year at 6% compound interest; what amount would he now have?

Ans., $15530.29, disregarding the extra day of the leap years.

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341. *Perpetuities.*

Ex. 1. Reckoning money as worth 5% annual interest, what is the present worth of an annuity of $650 to continue forever?

Evidently such a sum as put at interest will yield $650 annually; hence $\frac{650}{.05}$, or $13000.

It is interesting to note that the formula $P_w = \frac{(1+r)^t - 1}{r(1+r)^t} a$ gives the same result by making the *t infinite*,* and observing that $(1+r)^\infty$ is infinite, and hence that the -1 added to it can produce no effect upon it, and so can be dropped. Thus we have $P_w = \frac{(1+r)^\infty a}{r(1+r)^\infty}$, in which the factor $(1+r)^\infty$ cancels and leaves $P_w = \frac{a}{r}$, as above.

2. An annual income of £1000 in consols (***310***) corresponds to how much nominal present capital?

Ans., £33333 6*s.* 8*d.*

*** The symbol for an infinite quantity is ∞.**

3. What is the present worth of a perpetuity of \$2500, money being worth 4% annual interest?

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342. *Annuities in Reversion.*

Ex. 1. At 8%, what is the present worth of an annuity of \$450, to commence 5 years hence and continue thereafter 6 years? What if it continue forever thereafter?

Ans., \$1415.835; \$3828.285.

From (*340*) we learn that the present worth of \$1 annuity for 11 *yr.* at 8% is \$7.1390, and for 5 *yr.* \$3.9927. Hence the present worth of \$1 for the given time is \$3.1463. Again the present worth of a perpetual annuity of \$450 at 8% is \$5625, and for 5 *yr.* \$1796.715.

2. What is the present worth of an annuity of \$90, deferred 12 *yr.* and then to continue 7 *yr.*, at 4%?

Ans., \$337.39.

3. A father at his death left property to the amount of \$12000. He left as his heirs, his wife, and a son 11 *yr.* old. In his will he provided that a sufficient sum should be set apart to yield his son an annuity of \$1000 for 15 *yr.* after he became of age, compound interest 6%. The remainder of the estate he left to his wife for her own support and the support of the son till he became of age. What was the sum bequeathed to the wife? *Ans.*, \$6576.74.

4. A perpetuity of \$300 per annum is in reversion for 20 years. What is its present value, computed at 5% compound interest? *Ans.*, \$2261.34.

For time $T+t$ the P_w of an annuity is $\frac{(1+r)^{T+t}-1}{r(1+r)^{T+t}}a$, and for time t, $\frac{(1+r)^t-1}{r(1+r)^t}a$; hence for the present worth of an annuity deferred t years, and then to continue T years, is,

$$P_w = \left\{ \frac{(1+r)^{T+t}-1}{r(1+r)^{T+t}} - \frac{(1+r)^t-1}{r(1+r)^t} \right\} a.$$

In a similar manner, for a perpetuity in reversion for t years, we have $P_w = \left\{ \frac{1}{r} - \frac{(1+r)^t - 1}{r(1+r)^t} \right\} a = \frac{a}{r(1+r)^t}$.

5. Apply the last formula to find the present worth of a perpetuity of $360 in reversion 10 *yr.*, at 5%, using the compound interest table to find the value of $(1+r)^t$.

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343. *Annuities in Arrears.*

Ex. 1. What is the present worth of a perpetuity of $250 which is now in arrears for 10 *yr.*, reckoning 3% compound interest?

There is *now* due the *Amount* of $250 annuity at 3% for 10 *yr.* This is $11.4639 × 250. In addition to this there is the present worth of a perpetuity of $250, *i. e.*, $\frac{\$250}{.03}$. Total, $11199.31.

2. A father left his son, who was 15 years old, an annual allowance of $500 till he was 30; with the condition that he should receive the value thereof at 21, at 5% compound interest; provided the son's character at that age met the approbation of certain designated guardians. What should the son receive when he became of age if he fulfilled his part well?

Single Life Annuities.

(This is perhaps, with us, the most important case in Annuities, except those involved in Life Insurance, covering, as it does, an ordinary case of Widow's Dower.)

344. A Single Life Annuity is an annuity payable during the life of a specified individual; it is a *contingent annuity* dependent on the life of the individual.

345. A Temporary Life Annuity is an annuity payable for a specified number of years, provided the

person named lives through that period, but which ceases at the annuitant's death, if this occurs before the expiration of the time.

Of course, if it were possible to tell how long a *Life Annuitant* would live, the case would be one of annuity certain. But inasmuch as this cannot be, we are obliged to resort to the theory of probabilities, based upon as extensive observations as can be obtained upon the average duration of life, and the chances that a person at any given age will live till any other given age. Such tables are called *Mortality Tables*. Of these there are several in more or less general use, as, for example, the *Northampton*, the *Carlisle*, the *Combined Experience*, and the *American Experience*. We have space for but one, and hence give the latter as that in most general use in Insurance offices in this country, and legally recognized in several of the States, remarking, however, that the *Carlisle* has been more generally recognized than any other, especially by courts in this country. The above are all English tables except the last.

346. *American Experience Mortality Table.*

Age.	Survivors.	Deaths.	Age.	Survivors.	Deaths.	Age.	Survivors.	Deaths.
10	100,000	749	**30**	85,441	720	**50**	69,804	962
11	99,251	746	**31**	84,721	721	**51**	68,842	1,001
12	98,505	743	**32**	84,000	723	**52**	67,841	1,044
13	97,762	740	**33**	83,277	726	**53**	66,797	1,091
14	97,022	737	**34**	82,551	729	**54**	65,706	1,143
15	96,285	735	**35**	81,822	732	**55**	64,563	1,199
16	95,550	732	**36**	81,090	737	**56**	63,364	1,260
17	94,818	729	**37**	80,353	742	**57**	62,104	1,325
18	94,089	727	**38**	79,611	749	**58**	60,779	1,394
19	93,362	725	**39**	78,862	756	**59**	59,385	1,468
20	92,637	723	**40**	78,106	765	**60**	57,917	1,546
21	91,914	722	**41**	77,341	774	**61**	56,371	1,628
22	91,192	721	**42**	76,567	785	**62**	54,743	1,713
23	90,471	720	**43**	75,782	797	**63**	53,030	1,800
24	89,751	719	**44**	74,985	812	**64**	51,230	1,889
25	89,032	718	**45**	74,173	828	**65**	49,341	1,980
26	88,314	718	**46**	73,345	848	**66**	47,361	2,070
27	87,596	718	**47**	72,497	870	**67**	45,291	2,158
28	86,878	718	**48**	71,627	896	**68**	43,133	2,243
29	86,160	719	**49**	70,731	927	**69**	40,890	2,321

American Experience Mortality Table—Continued.

Age.	Survivors.	Deaths.	Age.	Survivors.	Deaths.	Age.	Survivors.	Deaths.
70	38,569	2,391	**80**	14,474	2,091	**90**	847	385
71	36,178	2,448	**81**	12,383	1,964	**91**	462	246
72	33,730	2,487	**82**	10,419	1,816	**92**	216	137
73	31,243	2,505	**83**	8,603	1,648	**93**	79	58
74	28,738	2,501	**84**	6,955	1,470	**94**	21	18
75	26,237	2,476	**85**	5,485	1,292	**95**	3	3
76	23,761	2,431	**86**	4,193	1,114	**96**	0	0
77	21,330	2,369	**87**	3,079	933	**97**	0	0
78	18,961	2,261	**88**	2,146	744	**98**	0	0
79	16,670	2,196	**89**	1,402	555	**99**	0	0

This table shows that of 100,000 persons living at the age of 10, 749 will die before they reach the age of 11, leaving 99,251; of these, 746 will die during the next year, leaving 98,505 survivors at the age of 12, etc. The table was constructed by Mr. Homans from the experience of the Mutual Life Insurance Company of New York, carefully compared with other statistics gathered in this country, and with the best European tables.

347. ***The Mathematical Probability*** of an event is the number of favorable opportunities divided by the whole number of opportunities. The *Mathematical Improbability* is the number of unfavorable opportunities divided by the whole number of opportunities.

ILL.—A man draws a ball from a bag containing 5 white and 2 black balls; the opportunities favorable to drawing a white ball are *five*, and the whole number of opportunities is seven; hence the mathematical probability of drawing a white ball is $\frac{5}{7}$. The mathematical improbability of drawing a white ball is $\frac{2}{7}$.

Ex. 1. According to the above table, what is the probability that a person who is 30 *yr.* old will live a year, *i. e.*, till he is 31?

The *favorable opportunities* are 84721, and the whole number of opportunities is 85441; *i. e.*, of 85441 who *may possibly live* the year

out it is assumed that 84721 *will certainly* live it out. Hence the probability is $\frac{84721}{85441}$.

2. What is the probability that a person who is 90 years old will live 3 years? What of one who is 50? Of one who is 16? *Ans.*, $\frac{79}{847}$, $\frac{66797}{69804}$, and $\frac{93362}{95550}$.

3. What is the probability that a person who is 50 years old will live one year? What that he will live to be 60? 70? 80?

4. What is the present worth of $300 due 1 year hence, contingent upon a person who is 40 years old living to the end of the year, money being reckoned worth 7%?

The present worth of $300 due 1 year hence at 7% is $280.3738. But the probability that a person who is 40 will live a year longer is $\frac{77341}{78106}$. Hence the present worth of the $300 on this condition is $\$280.3738 \times \frac{77341}{78106} = \$277.63$.

5. At 7%, what is the present worth of an annuity of $300 for 3 years, due a person who is 40 years old and contingent on his life?

As above, the $300 due in 1 year is worth $277.63.

The present worth of the $300 due in 2 *yr.*, *if certain*, would be $300÷1.1449, or $262.0317. But the probability of a man who is 40 living till he is 42 is $\frac{76567}{78106}$. Hence the present worth of this payment is $\$262{,}0317 \times \frac{76567}{78106} = \$256.87$.

In like manner, the present worth of the 3d payment, contingent on the annuitant's living 3 years, is $237.60.

Hence the entire present worth of the three payments is $277.63 + $256.87 + $237.60 = $772.10.

6. A widow at 80 has a right of dower (*i. e.*, $\frac{1}{3}$ interest) in an estate of $15000. What is its present worth at 5%?

The widow's annual income being 5% on $5000, is a life annuity of $250. Hence we have the following computation:

$$\text{Present worth of \$250 for 1 } yr. \times \frac{12383}{14474} = \$203.698$$

$$\text{Present worth of \$250 for 2 } yr. \times \frac{10419}{14474} = \$163.229$$

$$\text{Present worth of \$250 for 3 } yr. \times \frac{8603}{14474} = \$128.361$$

$$\text{Present worth of \$250 for 4 } yr. \times \frac{6955}{14474} = \$\ 98.83$$

$$\text{Present worth of \$250 for 5 } yr. \times \frac{5485}{14474} = \$\ 74.23$$

$$\text{Present worth of \$250 for 6 } yr. \times \frac{4193}{14474} = \$\ 54.043$$

$$\text{Present worth of \$250 for 7 } yr. \times \frac{3079}{14474} = \$\ 37.795$$

$$\text{Present worth of \$250 for 8 } yr. \times \frac{2146}{14474} = \$\ 25.088$$

$$\text{Present worth of \$250 for 9 } yr. \times \frac{1402}{14474} = \$\ 15.609$$

$$\text{Present worth of \$250 for 10 } yr. \times \frac{847}{14474} = \$\ 8.98$$

$$\text{Present worth of \$250 for 11 } yr. \times \frac{462}{14474} = \$\ 4.665$$

$$\text{Present worth of \$250 for 12 } yr. \times \frac{216}{14474} = \$\ 2.077$$

$$\text{Present worth of \$250 for 13 } yr. \times \frac{79}{14474} = \$\ 0.723$$

$$\text{Present worth of \$250 for 14 } yr. \times \frac{21}{14474} = \$\ 0.183$$

$$\text{Present worth of \$250 for 15 } yr. \times \frac{3}{14474} = \$\ 0.025$$

Answer, $817.536

For practical purposes tables are computed and are to be found in various lawyers' hand-books, as in those on Probate Practice, which give the present worth of $100 or $1000 dower or annual income, beginning at various ages and at several different rates of interest.

Those, howevet, which the author recollects to have noticed are reckoned on the basis of the Northampton or the Carlisle Tables. It would seem quite desirable that we have such tables computed on the basis of the most generally received American table of mortality, *i. e.*, the American Experience. There are practical expedients which somewhat shorten the work of computation as given above; but even the computation as thus given is not very laborious. Thus to make the above results correspond to $1000 annual income from dower, the several results should be multiplied by 4. Then, beginning at the last, the last would be the tabulated number for age 95; the sum of the two last, $0.208, for 94; the sum of the 3 last, $0.931, for 93, etc. Thus the above would give 15 years, which is about ⅛ of an ordinary table at 5%.

For the above computations the following table will be found convenient, although the results there given are more accurate than a 4-place table will yield:

348. Present Worth of $1 due at the end of any number of years, at Compound Interest.

Yrs.	3%.	3½%.	4%.	5%.	6%.	7%.	8%.	10%.
1	.9709	.9662	.9615	.9524	.9434	.9346	.9259	.9091
2	.9426	.9335	.9246	.9070	.8900	.8734	.8573	.8264
3	.9151	.9019	.8890	.8638	.8396	.8163	.7938	.7513
4	.8885	.8714	.8548	.8227	.7921	.7629	.7350	.6830
5	.8626	.8420	.8219	.7835	.7473	.7130	.6806	.6209
6	.8375	.8135	.7903	.7462	.7050	.6663	.6302	.5645
7	.8131	.7860	.7599	.7107	.6651	.6227	.5835	.5732
8	.7894	.7594	.7307	.6768	.6274	.5820	.5403	.4665
9	.7664	.7337	.7026	.6446	.5919	.5439	.5002	.4241
10	.7441	.7089	.6756	.6139	.5584	.5083	.4632	.3855
11	.7224	.6849	.6496	.5847	.5268	.4751	.4289	.3505
12	.7014	.6618	.6246	.5568	.4970	.4440	.3971	.3186
13	.6810	.6394	.6006	.5303	.4688	.4150	.3677	.2897
14	.6611	.6178	.5775	.5051	.4423	.3878	.3405	.2633
15	.6419	.5969	.5553	.4810	.4173	.3624	.3152	.2394
16	.6232	.5767	.5339	.4581	.3936	.3387	.2919	.2176
17	.6050	.5572	.5134	.4363	.3714	.3166	.2703	.1978
18	.5874	.5384	.4936	.4155	.3503	.2959	.2502	.1799
19	.5703	.5202	.4746	.3957	.3305	.2765	.2317	.1635
20	.5537	.5026	.4564	.3769	.3118	.2584	.2145	.1486

7. A widow at 85 has a dower interest in an estate of \$45000. What is its present worth at 4%?

Joint Life Annuities.

349. *A Joint Life Annuity* is an annuity contingent on the survival of all of several persons.

Ex. 1. What is the probability of both of two persons, one 30 and the other 40, living 5 years?

SOLUTION.—We find from the table that the probability of a person's living from 30 to 35 is $\frac{81822}{85441}$, and of one's living from 40 to 45 is $\frac{74173}{78106}$. Hence the probability that *both* will live the 5 years is $\frac{81822}{85441} \times \frac{74173}{78106} = \frac{606898}{667345}$, nearly.

2. What is the probability that both of two persons, one 36 and the other 50, will live 1 year? What of two, one 20 and the other 60?

3. What is the probability that all of a company of 5, whose ages are 16, 20, 21, 22, 23, will be alive at the end of 1 year?

4. What is the present worth of \$500 due 5 years hence, contingent of the lives of two persons, one 30 and the other 40, simple interest at 10%? What if money is worth 10% compound interest? (See Ex. 1.)

Ans., \$303.14; \$282.33.

5. What is the present worth of an annuity of \$200 for 3 years, contingent on the lives of *both* of 2 persons, one 40 and the other 50, compound interest at $3\frac{1}{2}\%$?

$\$193.24 \times \frac{77341}{78106} \times \frac{68842}{69804} + \$186.70 \times \frac{76567}{78106} \times \frac{67841}{69804} + \$180.38 \times \frac{75782}{78106} \times \frac{66797}{69804} = \$534.06$, *Ans.*

6. What is the present worth of a joint life annuity of $1000 to two persons, one ninety and the other 92, money being worth 4% compound interest? What at 4% simple interest?

By our table, this is the same as a joint annuity for 3 years.

7. How would you compute the present worth of a joint life annuity on two lives, one of which was 40 and the other 60 years old, at the commencement of the annuity?

Survivorships.

350. A Survivorship is an annuity contingent upon the survival of either of two or more lives.

351. The Improbability of an event is the difference between certainty and the probability, *i. e.*, it is 1—the probability; or it is the *unfavorable* opportunities divided by the whole number of opportunities.

Ex. 1. What is the present worth of $300 due 10 *yr.* hence, contingent upon *either* of two persons living through the 10 *yr.*, one of which persons is 36 years old and the other 50, interest 7%?

This case is resolved by observing that ***the Improbability*** *that both lives will fail during the period is the same as the* ***Probability*** *that the amount will be paid at the end of the time.* Thus if it is just as likely that both lives will fail as it is that both will continue, the improbability is $\frac{1}{2}$ and the value of the payment is $\frac{1}{2}$ of what it would be if certain. Again, if the improbability of the failure of both lives during the period is $\frac{3}{8}$, the probability that the $300 will be paid is $\frac{3}{8}$, etc.

Now from the table and by (***351***), we learn that the improbability that the first life will continue, or the probability that it will fail during the time is $\frac{7745}{81090}$; and, in like manner, the probability that the second will fail is $\frac{11887}{69804}$. Hence the *probability* that both will

fail is $\frac{7745}{81090} \times \frac{11887}{69804}$; and the *improbability* is $1 - \frac{7745}{81090} \times \frac{11887}{69804}$ $= .83735+$. Finally, the present worth of $300 *certain* at the end of 10 *yr.*, at 7% compound interest, is $152.49; whence the present worth contingent on the survival of one of the two lives is $152.49 × .83735 = $127.70.

2. What is the present worth of a life annuity of $100, contingent upon the survival of either of two lives, one 90 and the other 92, interest at 7%?

The present worth of the 1st payment is $\$93.46 \times \left(1 - \frac{385}{847} \times \frac{137}{216}\right)$; of the second, $\$87.34 \times \left(1 - \frac{631}{847} \times \frac{195}{216}\right)$; of the third, $\$81.63 \times \left(1 - \frac{768}{847} \times \frac{213}{216}\right)$. Here, according to our table, the older life terminates, and the other two payments are evaluated on the basis of the single life.* Thus the present worth of the fourth payment is $\$76.29 \times \frac{21}{847}$, and of the fifth, $\$71.30 \times \frac{3}{847}$. Total, $105.90.

Life Insurance.†

352. Life Insurance is the guaranteeing of money contingently on human life. [*Van Amringe.*]

353. As to the *Constitution of the Company*, Life Insurance companies may be *proprietary*, *mutual*, or *mixed*,—*Proprietary* when the stock is subscribed and the company constituted in the

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* If the formula is rigorously applied, it will give the same result; thus for the fourth payment we should have $\$76.29 \times \left(1 - \frac{826}{847} \times \frac{216}{216}\right) = \$76.29 \times \frac{21}{847}$; etc.

† Some would apply the term "insurance" when the contingency is loss of *property*, and "assurance" when it is loss of *life*. Another, and etymologically more warrantable distinction, is that the term "insurance" is to be applied to the guarantee *received*, and "assurance" to the guarantee *given*; thus I *insure* my life or property, and the company *assure* me or my heirs a certain sum in case of loss. But there seems to be no need of two terms, and American usage is to employ but one, viz., *insurance*.

ordinary way of organizing business corporations; *Mutual* when each person insured becomes a member of the company, and hence is both insured and insurer; *Mixed* when both features are combined.

354. *Policies* are *Life Policies* when the amount guaranteed is due on the death of the insured; *Term Policies* when this sum is payable upon the death of the insured, provided it occurs within a specified time; *Endowment Policies** when the guarantee is payable when the insured reaches a certain age, or at his death if it occurs before he reaches that age. (See ***361.***)

355. Theoretically an amount insured contingent on the death of a person is payable at the close of the year within which the death occurs; nevertheless it is customary for good companies to pay within sixty or ninety days after the necessary proofs of death have been filed.

356. *Premiums* are *Single* when the insurer pays in advance the entire sum necessary to secure the payment of the special sum at death; *Limited* when the entire premium is to be paid in a specified number of payments; and *Annual* when the insurer pays a stipulated sum annually during the lifetime of the insured.

357. The rate of premium as compared with the amount insured depends upon three principal considerations, viz.: 1. The probability of the duration of the life of the insured; 2. The probable rate of interest which the company can secure on the premiums paid in; 3. The estimated expenses of the company in carrying on the business.

In *Mutual Companies* the rate of interest which premiums are calculated to earn for the company is usually put at 4%; in proprietary companies it is generally somewhat higher.

358. Net Premium is the premium which the first two considerations—probability of life and interest on premiums—would demand.

* This is the common use of the term in this country.

359. Loading is the amount added to the net premium to cover the estimated expenses of the company, and to provide for unusual mortality or other exigencies which cannot be foreseen.

Ex. 1. What is the *net single premium* required to insure a life at 90 for $100, interest 4%?

This is the same as the present worth of $100 to be paid at the death of a person now 90 years old, interest at 4%.

If death were *certain* at the end of the first year, this would be the present worth of $100 at 4% for 1 *yr.*, *i. e.*, $96.15 (*348*).

But the *probability of death* during this year (*346*) is $\frac{385}{847}$; hence the present worth on this contingency is . $\$96.15 \times \frac{385}{847} = \$43.704$

In like manner, if the $100 had *certainly* to be paid at the end of 2 *yr.*, its present worth would be $92.46; but, subject to the contingency of the death of the person during these 2 years, it is worth . $\$92.46 \times \frac{246}{847} = \$26.853$

In like manner for the next year we have $\$88.90 \times \frac{137}{847} = \$14.379$

For the next, ... $\$85.48 \times \frac{58}{847} = \$\ 5.853$

For the next, ... $\$82.19 \times \frac{18}{847} = \$\ 1.746$

For the next, ... $\$79.03 \times \frac{3}{847} = \$\ .29$

Hence the net premium is $92.83

2. What is the *net single premium* required to insure a life at 75 for $3000, interest 5%? *Ans.*, $2198.20.

3. From the above let the student show how he would proceed to find the net single premium to insure a life at 40 for $2500, interest 4%.

The actual computation need not be attempted, as it is exceedingly tedious (in fact, our table of present worth (*348*) is not suf-

ficiently extended for this case). But a clear conception of the principles and the process can be obtained from the above.

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4. What is the *annual net premium* to be paid during life in order to insure a life at 90 for $100, interest at 4%? (See Ex. 1.)

This annual premium is evidently *an annuity*, the present worth of which is what would be required as a net single premium, that is, $92.83. (See Ex. 1.)

Now $1 paid down is worth $1.00

$1 paid at the end of the first year, or when the insured is 91 *yr.* old, would be worth $0.9615 if it were certain that the insurer would survive to pay it. But the contingency is $\frac{462}{847}$ of certainty; hence the present worth of $1 subject to this contingency is . . . $\$0.9615 \times \frac{462}{847} = \$0.5244$

So $1 at the end of 2 *yr.* is worth . . . $\$0.9246 \times \frac{216}{847} = \$0.2357$

$1 " " 3 *yr.* " " . . . $\$0.889 \times \frac{79}{847} = \$0.0829$

$1 " " 4 *yr.* " " . . . $\$0.8548 \times \frac{21}{847} = \$0.0212$

$1 " " 5 *yr.* " " . . . $\$0.8219 \times \frac{3}{847} = \$0.0029$

The probability that the person will be alive at the end of the next year being 0, we have the total present worth of $1 annual premium, ... $1.8671

Hence the net annual premium is $92.83 ÷ $1.8671 = $49.72.

5. What is the *annual net premium* to be paid during life in order to insure a life at 75 for $3000, interest 5%? (See Ex. 2.) *Ans.*, $393.06.

6. What is the net annual premium to be paid at the beginning of each of the first 10 years in order to secure a policy of \$3000 to a person aged 75, interest 5%?

Divide the *net single* premium by the sum of the present worths of \$1 paid at beginning of each of the first 10 years, subject to the contingency of the insured's death, as in Ex. 4. This gives \$2198.20 ÷ 5.21 = \$421.91.

7. From the above let the student show how he would proceed to find the *annual net premium* to be paid during life in order to insure a life at 35 for \$5000, interest at $4\frac{1}{2}\%$. Also the annual premium if *all be paid* in 20 payments.

8. What is the *net single premium* to be paid for a term policy of 5 years, of \$3000, at the age of 30, interest at 4%?

We will first find the premium for \$1 insured.

If this was certainly to be paid at the end of the first year, its present worth would be \$0.9615; but the probability that the person will die, and hence that it will have to be paid, is $\frac{720}{85441}$; hence we have . $\$0.9615 \times \frac{720}{85441} = \$0.00810$

In like manner for the 2d year, . . $0.9246 \times \frac{721}{85441} = 0.00780$

For the 3d year, ... $0.889 \times \frac{723}{85441} = 0.00752$

" 4th " . . . . $0.8548 \times \frac{726}{85441} = 0.00726$

" 5th . . . . $0.8219 \times \frac{729}{85441} = 0.00701$

Total to insure \$1 for 5 years, ... \$0.03769

Therefore the *single* premium for \$3000 is \$113.07.

9. What is the *net annual premium* to be paid for a term policy of 5 years, of \$3000, taken out at the age of 30, interest 4%?

In the last example we found the *single* premium to be $113.07. We have now to find the present worth of $1 paid at the commencement of each of the 5 years, contingent on the survival of the insured, that is:

Present worth of $1 paid down, . . .		$1.00
" " " $1 " at the beginning of the 2d year, ...	$\$0.9615 \times \frac{84721}{85441} =$	0.9533
Ditto at beginning of 3d year, ...	$0.9246 \times \frac{84000}{85441} =$	0.9090
" " " " 4th " . . .	$0.889 \times \frac{83277}{85441} =$	0.8665
" " " " 5th " . . .	$0.8548 \times \frac{82551}{85441} =$	0.8259
Total present worth of $1 annual premium, ...		$4.5547

Hence the annual premium required is $113.07 ÷ $4.5547 = $24.83.

10. Let the student show how to compute the *net annual premium* on a $5000 10-year policy, taken out at the age of 40, interest 5%.

360. Loading. (See ***359.***) This is usually reckoned at a certain per cent. on the net premium as determined above. It is different on different kinds of policies, and varies slightly in different companies. From 20% to 40% is the common range for loading, the average being about 33⅓%. The net premium plus the loading makes the ***Office Premium,*** which is what the insurer actually pays.

11. Find the *Office Premium* in examples 1, 2, 4, 5, 8, 9, at 20%, 25%, and 40%.

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361. Endowment (Simple) is a contract which guarantees the payment of a certain sum at a specified future time, contingent upon the assured's living till that time.

12. What is the *net simple premium* required to secure to a person 40 years old $2000 when he reaches 50, interest at 4%?

$1 *certain* 10 *yr.* hence is worth .6756. But the probability of a person at 40 living to 50 is $\frac{69804}{78106}$. Hence the present worth of $1 due 10 *yr.* hence, and subject to this contingency, is $.6756 \times \frac{69804}{78106} = .60378$; and of $2000, $1207.56.

13. What must be the *net annual premium* to insure $2000 endowment to a man at 50 who is now 40, interest at 4%?

According to principles heretofore developed, this is evidently $1207.56 (see above) divided by the sum of the present worths of $1 payable at the beginning of each of the 10 years, each contingent upon the man's being alive to pay it. This is $1207.56 ÷ 8.0758 = $149.53.

14. Let the student describe the process of finding the *Office Annual Premium* required to secure to a man at 40 $15000 if he should reach 60, interest at 5%, loading 25%.

As the operation is exceedingly tedious, it will be sufficient for most students if they *indicate clearly all the steps in the process*, and comprehend them. All the work should be written out in form, but the actual multiplications and divisions need not necessarily be performed.

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362. Endowment Insurance, as usually understood in this country, is a contract to pay a certain sum when the insured reaches a certain age, *or at his death*, if it occurs before he reaches that age (*354*).

15. What *net annual premium* is required to secure an *Endowment Policy* (*262*) guaranteeing to a man at 40 $5000 when he reaches 50, or the same to his heirs at his death, if he should die before that time, interest 4%? What the *office premium*, allowing 25% for loading?

1. As in other cases, first find the *net single premium* required. This consists of two parts: (*a*) the *net single premium* as of a *simple*

endowment, and (*b*) the *net single premium* as of a *term endowment*. Thus if the man live to 50, the company will *certainly* have to pay the $5000 at that time. Now as in Ex. 11, we find the net single premium for this risk to be $3018.90. But *in addition to this*, the company runs the risk of having to pay the $5000 at the end of any one of the 10 years. This, computed as in Ex. 8, is found to be $428.10. Hence *the net single premium required is* $3018.90 + $428.10 = $3447.

2. To find the *net annual premium*, as before, we divide the net single premium by the sum of the present worths of $1 paid at the beginning of each of the 10 *yr.*, subject to the contingency of the man's living. This in Ex. 12 was found to be $8.0758. Hence the net annual premium required is $3447 ÷ 8.0758 = $426.83.

3. The *office premium* is therefore $426.83 × 1¼ = $533.54 annually.

16. Let the student describe the process of finding the *office annual premium* required to secure to a man at 30 the payment of $10000 when he reaches 60, or to his heirs if he die before that age is reached, interest at 5%, loading 30%.

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363. Surrender. It has become the custom of many insurance companies to allow policy-holders to surrender them, that is, to pay the holder a small sum and take up the policy. Perhaps there is nothing in the practice of life insurance that more astonishes the uninitiated than the smallness of the sum which a company will pay for the surrender of a policy. It is a very natural view of the insured that he should receive at least the sum of his premiums, giving the company the advantage of the interest thereon for the time that has elapsed since they were paid. But no company would think of any such thing. The surrender value of a $1000 policy taken out at 30 years of age would, at the end of ten years, be less than $100, although the insurer would have paid about $25 annually, or in all $250. What is called the *Non-Forfeiting* plan is becoming quite common; that is, the practice of giving to the insured who may wish to surrender a policy a small "paid-up" policy. This guarantees to the insurer or his heirs a certain sum at the death of the insured without further payment on his part. Of course a much larger sum can be guaranteed in this way than

could be paid in cash. Our space does not allow us to exhibit the method of calculating these "surrender values," and in fact there is a variety of methods in use. The best service we can render *Teachers* and others who would become more broadly intelligent on this interesting subject is to commend to them "*A Plain Exposition of the Theory and Practice of Life Assurance,*" a pamphlet of 61 pp. by Prof. J. H. Van Amringe of Columbia College, New York City, 1874; published by Chas. A. Kittle. If more extended knowledge is desired, a little book written by Actuary *Nathan Willey*, and published by J. H. & C. M. Goodsell, will be found serviceable. Of course proficients will have access to such books as "Jones on Annuities," "Bailey on Annuities," the articles in the larger Encyclopædias, and the Insurance reports, especially those of Massachusetts and New York.

364. Expectation of Life. By this phrase is meant the probable number of years which a person will live after any given age, based upon the mortality tables. Thus, according to our table (***346***), the expectation of a man at 40 is 28.2 *yr.*, at 50 20.9 *yr.*, at 60 14.1 *yr.* This element, though of interest to the general reader, is of no practical use in insurance computations. The expositions often given on this basis are entirely erroneous; no such method is ever used.

SECTION V.

PARTNERSHIP.

[The ordinary definitions given under this head will be found in other places in this book, and the terms will be familiar to the student before he reaches this point. Indeed the problems are so simple as to need no further exposition; yet a few are inserted in accordance with custom, and to gratify those who might think the omission a serious oversight.]

Simple Partnership.

365. Simple Partnership is a partnership in which the capital of each partner is employed for the same time.

Ex. 1. A, B, and C enter into partnership; A puts in \$120, B \$200, C \$160; they gain \$96; what is each man's gain?

SOLUTION.

$$\begin{array}{l} \$120 \\ \$200 \\ \$160 \\ \hline \$480 \end{array} \therefore \left\{ \begin{array}{l} \text{A owns } \frac{120}{480}=\frac{1}{4} \text{ of capital.} \\ \text{B owns } \frac{200}{480}=\frac{5}{12} \text{ of capital.} \\ \text{C owns } \frac{160}{480}=\frac{1}{3} \text{ of capital.} \end{array} \right. \therefore \left\{ \begin{array}{l} \text{A's share is } \frac{1}{4} \times \$96=\$24. \\ \text{B's share is } \frac{5}{12} \times \$96=\$40. \\ \text{C's share is } \frac{1}{3} \times \$96=\$32. \end{array} \right.$$

Such examples may also be solved by percentage; thus, in this case, A owns 25% of the capital, and hence should have 25% of the profits; B for a like reason should have 41⅔% of the profits; and C 33⅓%.

Again the solution can be effected by simple proportion; thus,

The whole capital is to *A's capital* as *the whole gain* is to *A's gain*.

$$\$480 \quad : \quad \$120 \quad :: \quad \$96 \quad : \quad x \ (=\$24).$$

2. A, B, C, and D enter into partnership with a paid-in capital of \$10000, of which A puts in \$3500, B \$2500, C \$2500, and D \$1500. D is to manage the business, and his services are to be reckoned as equivalent to \$5000 capital. At the close of a year they find the net profits to be \$4500. What is each man's share?

3. A ship valued at \$24840 was owned jointly by three persons, P, Q, and S. For the building of the ship P contributed \$8280, Q \$4968, and S the remainder. Having been injured in a storm she put into a harbor for repairs. The expense of repairing was \$9315; how should this expense be apportioned among the three owners?

Ans., P, \$3105; Q, \$1863; S, \$4347.

4. In a certain business A received \$100 of the profits, B twice as much, C four times as much, and D as much as the other three. If the whole capital was \$2800, how much did each put in?

5. A bankrupt's assets are found to be \$4500 and his debts \$115000. What will each of three creditors receive

to whom he is indebted \$650, \$400, and \$840 respectively, provided it costs \$600 to settle the business.

6. Mr. A, who has failed in business, owed me \$250. His assets were \$5500, and I received \$100. What were his liabilities? What would a man receive to whom A owed \$88?

[For other examples usually found under this head see *Partitive Proportion*, p. 135.]

Compound Partnership.

366. Compound Partnership is a partnership in which the capital of the partners is employed for different periods of time.

Ex. 1. A, B, and C enter into partnership; A puts in \$100 for 4 months, B \$300 for 2 months, and C \$500 for 3 months; they gain \$250. How much is each man's gain?

SOLUTION.—The use of \$100 4 *mo.* = the use of \$400 1 *mo.*
" " \$300 2 *mo.* = " " \$600 1 *mo.*
" " \$500 3 *mo.* = " " \$1500 1 *mo.*
Therefore the use of the entire capital for the respective times equals the use of . . . . \$2500 1 *mo.*

Hence A should have $\frac{400}{2500}$, $\frac{4}{25}$, or 16% of the gain; B, $\frac{600}{2500}$, $\frac{6}{25}$, or 24%; and C, $\frac{1500}{2500}$, $\frac{3}{5}$, or 60%.

2. A, B, and C trade in company. A puts in \$300 for 5 months, B puts in \$400 for 8 months, and C puts in \$500 for 3 months; they gain \$100. What is the gain of each?

3. Three men rented a pasture, agreeing to pay \$38.40 for its use. The first put in 18 cows for 3 months, the second 24 cows for 4 months, and the third 12 cows for $3\frac{1}{2}$ months. How much of the rent ought each to pay?

4. Four men performed a piece of work, for which they

received $1200. The first worked 25 *da.* of 10 *hr.* each, the second 110 *da.*, 8 *hr.* each, the third 108 *da.*, 9 *hr.* each, and the fourth 200 *da.*, 10 *hr.* each. What amount should each receive?

5. A and B entered into partnership for 2 years. At first, A furnished to the joint capital $2500; and at the end of 8 months he put in $1000 more. B at first put in $3000, and at the expiration of 9 months took out $1000. They gained $877.45 by trading. How ought this gain to be divided?

6. The joint stock of a company was $5400, which was doubled at the end of the year. A put $\frac{1}{3}$ for $\frac{3}{4}$ of a year, B $\frac{2}{5}$ for $\frac{1}{2}$ a year, and C the remainder for one year. How much was each one's share of the entire stock at the end of the year?

7. Three men took an interest in a coal mine. B invested his capital for 4 months, and claimed $\frac{1}{10}$ of the profits; C's capital was in 8 months; and D invested $6000 for 6 months, and claimed $\frac{4}{5}$ of the profits. How much did B and C put in?

8. Three men engaged in merchandising. A's money was in 10 months, for which he received $456 of the profits; B's was in 8 months, for which he received $343.20 of the profits; and C's was in 12 months, for which he received $750 of the profits. Their whole capital invested was $14345. How much was the capital of each?

9. Three merchants are concerned in a steam-vessel; the first, A, put in $960 for 6 months; the second, B, an unknown sum for 12 months; and the third, C, $640, for a time not known. When the accounts were settled A received $1200 for his stock and profit, B $2400 for his, and C $1040 for his. What were B's stock and C's time?

CHAPTER VIII.

367. PROBLEMS FOR REVIEW.

The more special design of this chapter is to afford a final review of certain subjects treated with sufficient fullness in the Elements, and hence not rediscussed in the preceding chapters, and to give a few additional problems in *Mensuration*.

1. How many acres in a rectangular piece of ground 80 rods by 140?

2. How many acres in a circle whose radius is $\frac{1}{2}$ a mile? What is the circumference of such a circle?

3. When it is 12 o'clock M. at Rochester, N. Y., it is 9 *hr.* 1 *min.* 37 *sec.* A. M. at San Francisco. The *long.* of Rochester being 77° 51′ W., what is the *long.* of the latter?

4. In the latitude of Ann Arbor, Mich., 51.1 *mi.* make a degree of longitude. How many miles west of Ann Arbor is Milwaukee, Wis., the difference in time being 16 *min.* 12$\frac{3}{4}$ *sec.*?

5. In reducing longitude to time, why does multiplying degrees, minutes, and seconds by 4 give respectively minutes, seconds, and 60ths of seconds? Why does multiplying hours, minutes, and seconds of time by 15 give respectively degrees, minutes, and seconds of longitude?

6. Saturn makes a revolution on its axis in about 10$\frac{1}{2}$ hours. How many degrees of longitude correspond to an hour of time if an hour is $\frac{1}{24}$ of its day? How many degrees correspond to 1 hour of our time? If its equatorial diameter is 75000 miles, how many miles on its equator make an hour of longitude?

7. If my watch is right with Ann Arbor time (83° 50′ 48″.3), and keeps correct time, how should it compare with Boston time (71° 3′ 30″)? How with Madison, Wis. (89° 23′)?

8. A telegraphic signal sent from Ann Arbor at 10 o'clock 15 *m.* 20 *sec.*, is received at St. Paul, Minn., 59 *m.* 43⅛ *sec.* before noon. What is the longitude of St. Paul? (See last example.)

9. In the centigrade thermometer the freezing point is zero, and the boiling point is 100°; in Fahrenheit's the freezing point is 32°, and the boiling point is 212°. What degree of the centigrade thermometer corresponds to the 68th degree of Fahrenheit?

10. A room contains 432 square feet, and its breadth is to its length as 3 to 4. What are the dimensions of the room?

11. A straight plank is 3¼ inches thick and 6¼ inches broad. What length must be cut off so that the part cut off may contain 6¼ cubic feet of timber?

12. A watch which loses 4 minutes a day was set right at 12 o'clock on April 10th. What will be the true time on April 20th, when the hands of the watch point to 12 o'clock?

13. When the days and nights are of equal length, and it is noon on the first meridian, on what meridian is it then sunrise? Sunset? Midnight?

14. A and B start at the same time from opposite points of a circular park 864 feet in circumference, and walk in the same direction. A goes 144 feet, and B 156 feet, in a minute. In what time will B overtake A, and how far will each have walked when A is overtaken by B?

15. A farmer bought a pile of wood 40 *ft.* long, 12 *ft.*

wide, and 10 *ft.* high. at \$4¼ per cord. What did he pay for his wood?

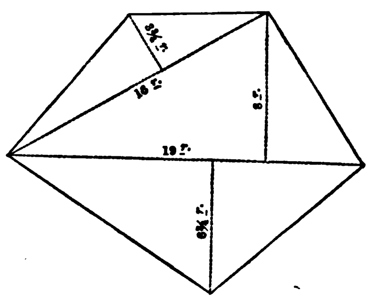
16. What must be the length of a plot of ground, if the breadth be 15¾ feet, that its area may contain 46 square yards?

17. How many yards of carpeting that is ¾ of a yard wide will be required to carpet a room that is 35 feet long and 18 feet wide; and what will the carpeting cost at \$1.75 a yard?

18. When it is noon at Detroit (82° 58′), on what meridian is it midnight? On what 6 A. M.? On what 6 P. M.?

19. A brick is 8 inches long, 4 inches wide, and 2 inches thick. How many such bricks will be required to build a cubical cistern, open at the top, that shall contain 2000 wine gallons, if the wall is made a foot thick, and we allow that the mortar in which the bricks are laid will constitute a fifth part of the wall?

20. Knowing that the area of a triangle is ½ the product of its base and altitude, find the number of acres included in the field represented in the accompanying diagram, the lengths of the sides, the diagonals, and the altitudes of the triangles having been found by measurement to be as given in the figure.



21. A watch which gains 2 minutes a day was set right at 12 o'clock M., on the 24th of March. What will be the

true time on March 30th, A. M., when the hands are in a right line, and the hour hand is between 9 and 10?

22. What length of brace is required for a gate 5 *ft.* high and 11 *ft.* long, the brace to be a diagonal?

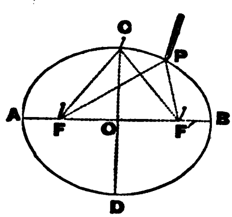
23. A boy standing 5 rods from the root of a tree shot at a squirrel on the top of the tree, directly over the base, and 60 *ft.* from the ground. What was the distance?

24. How many barrels does a cistern contain which is a circular cylinder 7 *ft.* in diameter and 8 *ft.* deep?

25. The area of the surface of a sphere is equal to the area of 4 of its great circles.* What is the area of the surface of the earth, its radius being 3960 miles?

26. The volume of a sphere being $\frac{1}{3}$ the product of its radius into its surface, what is the volume of a sphere whose diameter is 12 *in.*? What the volume of the earth in cubic miles?

368. ***An Ellipse*** is an oval figure which may be described by taking a string (FPF′) as long as it is desired to have the *Transverse Axis* (AB), and placing a pin at C, the extremity of the *Conjugate Axis* (CD), pass the string around it, and fasten the ends at two points in the transverse axis, F and F′, so that CF = CF′. Then remove the pin at C, and putting a pencil, P, in its place, carry the pencil around, keeping the string tight. F and F′ are the Foci.



Calling AB = $2a$, and CD = $2b$, the area of an ellipse is πab.†

27. How would you lay out an elliptical garden bed 10 *ft.* long and 6 *ft.* wide? What would be its area?

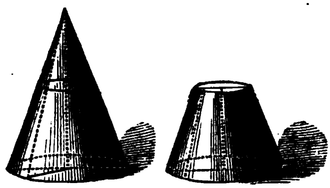
* A great circle of a sphere is the circle which passes through the center—it is the base of the hemispheres.

† π is the Greek letter *pi*, and in mathematics stands for 3.1416 (approximately), *i. e.*, the ratio of the circumference of a circle to its diameter.

28. What is the area in acres of an ellipse whose transverse axis is 100, and its conjugate 60 rods?

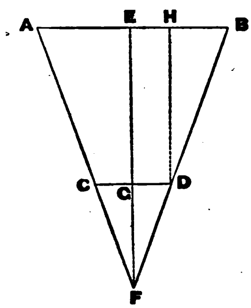
QUERY.—If you attempt to construct an ellipse whose axes are equal, what is the result? Where will the foci be? Will the formula πab give its area?

29. Knowing that the volume of a cone is $\frac{1}{3}$ the product of the area of its base into its altitude, what is the volume of a cone the radius of whose base is 6 *ft.* and whose altitude is 4 *ft.*?



CONE. FRUSTUM OF CONE.

30. What are the contents in barrels of a cistern in the form of an inverted frustum of a cone, the diameter of the bottom being 4 *ft.*, that of the top 10 *ft.*, and the depth 8 *ft.*?



The first thing is to find the entire altitude of the complete cone, that is, EF. This is done on the principle that the corresponding sides of similar triangles (those of the same shape) are proportional. Now BHD and BEF are similar triangles; and as HB = 3 *ft.*, EB = 5 *ft.*, and HD = 8 *ft.*, we have EF = $\frac{5}{3}$ HD, or $13\frac{1}{3}$ *ft.* Hence GF = $13\frac{1}{3} - 8 = 5\frac{1}{3}$ *ft.*

31. What will it cost to put a walk around the outside of a square park that contains 10 acres, if the walk is 4 feet wide, and the price of paving is $6\frac{1}{4}$ cents per square foot?

32. How many pints does a coffee-pot contain which is $4\frac{1}{2}$ *in.* across the top, 7 *in.* across the bottom, and $8\frac{1}{2}$ *in.* deep?

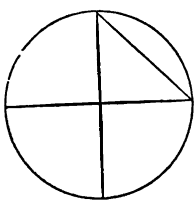
33. Knowing that the convex surface of a cone is $\frac{1}{2}$ the

product of its slant height into the circumference of the base, what is the area of the wall and bottom of the cistern (Ex. 30)?

34. The mean diameter of a barrel being equal to the head diameter + $\frac{2}{3}$ the excess of the bung diameter over the head diameter,* how many gallons does a barrel contain whose measurements are, head diameter 17 *in.*, bung 20 *in.*, length 28 *in.*? How many one whose measurements are 19 *in.*, 23 *in.*, and 31 *in.*?

Finding the contents of casks from proper measurements is called *Gauging.* But if any one supposes that such a rule as the above disposes of the matter, he is greatly in error. For accurate gauging, as in custom-house service, carefully prepared special instruments and extensive and accurate tables are requisite. The above gives simply a near approximation.

35. What is the largest square stick of timber which can be cut from a log 36 *in.* in diameter?



36. How large a log is required to secure a timber 15 *in.* square?

37. Show that the diameter of a circle multiplied by .7071 + gives the side of the greatest square (called the inscribed square) which can be cut from the circle.

369. Lumbermen usually take $\frac{2}{3}$ of the *mean diameter* of a log as the side of the square timber which can be hewn from it. This, it will be seen, makes the square timber a little smaller than mathematical accuracy requires. Nevertheless it is *practically* nearer correct, inasmuch as logs are neither perfect cylinders nor perfectly straight.

* This when the staves are considerably curved; if they are pretty straight, $\frac{3}{5}$ is nearer correct.

38. According to the above, what are the cubic contents of hewn timber yielded by a log 24 *ft.* long and whose end diameters are 20 *in.* and 28 *in.*?

The *mean diameter* is $\frac{20+28}{2} = 24$ *in.*; and the side of the square timber 16 *in.*

39. Show that *twice the square* of the *radius* of a circle is the *area of the inscribed square.*

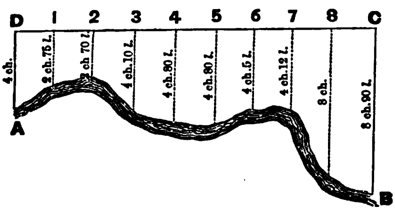
40. What is the greatest cube which can be cut from a sphere 40 *in.* in diameter?

41. What is the length of an arc containing 75 degrees on the circumference of a circle whose radius is 5 feet?

42. How many *degrees* are there in an arc of 3 feet in length on the circumference of a circle of radius 10 feet?

43. What amount of water would be discharged through a 2 *in.* cylindrical pipe in 10 *hr.*, the flow being 20 *ft.* per minute?

44. Having an irregular field as represented in the cut, the side DC was measured and found to be 19 *ch.*, and the ends AD and BC, 4 *ch.* and 8 *ch.* 90 *l.* Then the width was measured at 8 intermediate equidistant points along the line DC, and found to be 2 *ch.* 75 *l.*, 2 *ch.* 70 *l.*, 4 *ch.* 10 *l.*, 4 *ch.* 80 *l.*, 4 *ch.* 80 *l.*, 4 *ch.* 5 *l.*, 4 *ch.* 12 *l.*, and 8 *ch.* How many acres in the piece?



The average width is the sum of the intermediate measurements $+\frac{1}{2}$ the sum of the extremes, divided by the number of measurements $-$ 1.

45. Wishing to ascertain the amount of water discharged by a certain river, a place was selected where the banks were nearly regular slopes and the bottom level, making the river of uniform depth. Measurements were then taken which showed that the stream was 351 *ft.* wide on the bottom, 359 *ft.* on the surface, and $5\frac{1}{2}$ *ft.* deep. A float on the stream showed that the current flowed 2 *mi.* per hour. How many barrels would such a river discharge in 24 hours?

46. The depth of a river 560 *ft.* wide was measured at six equidistant places in a line directly across it, and found to be 2 *ft.*, 3 *ft.*, 4 *ft.*, $4\frac{1}{2}$ *ft.*, 2 *ft.* $1\frac{1}{2}$ *ft.*, and being very shallow at the margins, the depths were called 0. Five measurements of the current along the same line showed it to be $\frac{1}{5}$ *mi.*, $\frac{1}{2}$ *mi.*, 1 *mi.*, $\frac{1}{4}$ *mi.*, and $\frac{1}{10}$ *mi.* per hour. How many barrels of water would the river discharge in 24 hours?

The average depth was $2\frac{3}{7}$ *ft.*, and the average rate of current, $.34\frac{1}{6}$ *mi* Why?

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370. ***Doyle's Rule*** for calculating the amount of square-edged inch boards which can be sawed from a round log is this: *From the diameter in inches subtract 4; the square of the remainder will be the number of square feet of inch boards yielded by a log 16 ft. in length.*

The yield of logs of the same diameter is in the ratio of their lengths.

Ex. 1. How much square-edged inch lumber can be cut from a log 28 *in.* in diameter and 12 *ft.* long?

Ans., 432 *ft.*

$\frac{3}{4}$ of $24 \times 24 = 18 \times 24 = 432$.

2. What is the yield of a log 36 *in.* in diameter and 18 *ft.* long? Of one 36 *in.* in diameter and 15 *ft.* long? 20 *ft.* long?

3. What is the yield of a log 44 *in.* in diameter and 10 *ft.* long? 24 *ft.* long?

4. What is the yield of a log 12 *in.* in diameter and 12 *ft.* long? 18 *ft.* long? 16 *ft.* long?

5. What is the yield of a log 15 *in.* in diameter and 16 *ft.* long? 11 *ft.* long? 18 *ft.* long? 30 *ft.* long?

This rule, so admirable in its simplicity, is the foundation of the table in *Scribner's* popular *Lumber and Log Book*, which is said to have a larger sale than all other books of the kind together, and is a generally recognized standard among lumbermen. Nevertheless, in a scientific point of view, the rule is but a rude approximation, favoring the buyer in the case of small logs and the seller in the case of large ones, while for average logs (say from 18 *in.* to 30 *in.* in diameter) it gives a sufficiently just approximation. For a log 14 *in.* in diameter the rule gives the same result as squaring the log and allowing $\frac{1}{4}$ for saw kerf; while for a log 35 *in.* in diameter it gives $\frac{3}{4}$ the solid contents, *i. e.*, is equivalent to making no allowance for slab and $\frac{1}{4}$ for kerf. For a log 50 *in.* in diameter the rule is equivalent to allowing nothing for slab and less than $\frac{1}{8}$ for kerf. From a perfectly straight, cylindrical log 24 *in.* in diameter and 16 *ft.* long it is possible to saw 437 square feet of square-edged boards, allowing $\frac{1}{4}$ *in.* for saw kerf, and edging the boards separately. Doyle's rule gives 400 *ft.* for the measure of such a log. But considering the facts that logs are not perfectly straight cylinders, and that the log is slabbed in sawing, Doyle's rule gives a more just result than the theoretical computation.

APPENDIX.

TABLES OF DENOMINATE NUMBERS.

Measures of Extension.

370. ***The Fundamental Table*** is the table of *Linear* or *Long Measure*. From this all others are deduced. (See ***40, 41.***) The *Standard Unit* of any measure is the denomination which is particularly established by law, and of which the others are made certain multiples or parts. The U. S. *Standard of Comparison* for all measures of extension is a brass rod 82 *in.* long, obtained from England. This rod is to be used for comparison at 62° Fah. (***40***).

LINEAR MEASURE.	SQUARE MEASURE.	CUBIC MEASURE.
12 *in.* = 1 *ft.* ∴	144 *sq. in.* = 1 *sq. ft.*, and	1728 *cu. in.* = 1 *cu. ft.*
3 *ft.* = 1 *yd.*	9 *sq. ft.* = 1 *sq. yd.*, and	27 *cu. ft.* = 1 *cu. yd.*
$5\frac{1}{2}$ *yd.* = 1 *rd.* ∴	$30\frac{1}{4}$ *sq. yd.* = 1 *sq. rd.*	128 *cu. ft.* = 1 *cd.*
320 *rd.* = 1 *mi.*	160 *sq. rd.* = 1 *A.*	
	640 *A.* = 1 *sq. mi.*	

For the proposed theoretical method of restoring the English yard by comparison with the pendulum, in case of loss, see ART. ***40.*** On this it is to be remarked that no standard was ever so obtained, and that when the English standard of comparison was lost by the burning of the Parliament House in 1834, the standard was restored on quite other principles. In this country it does not appear that Congress has made any provision for the restoration of the standard of comparison in case of the loss of the Troughton scale referred to above. In such an event, however, there is little doubt that it would be restored from the copies in possession of the States.

On the common Carpenter's Square the inch is usually divided into *halves, quarters, eighths,* and *sixteenths,* or into *twelfths.* On other scales it is often divided into *tenths.* *A Line* is $\frac{1}{12}$ of an inch.

A *Size*, as used by shoemakers, is $\frac{1}{3}$ of an inch. *Children's sizes* run from *size* 1, $4\frac{1}{3}$ *in.* long, to size 13, $8\frac{1}{3}$ *in.* long. *Youth's, women's,* and *men's sizes* run from *size* 1, $8\frac{1}{2}$ *in.*, to *size* 15, $13\frac{1}{6}$ *in.*

The ancient *Roman Mile* (*mille passum,* 1000 paces) was about 1618 *yds.*, and hence a little shorter than ours. The modern *Ro-*

man Mile = .925 *Eng. mi.* The *Irish Mile* = 1.273 *Eng. mi.* The *French Mile* (*mille marin*) is the same as our marine or geographic mile. The *German Short Mile* (*meile*) = 3.897 *Eng. mi.*; the *Long Mile* = 5.753 *Eng. mi.*; the *Prussian Mile* = 4.68 *Eng. mi.*

The *Equatorial Diameter* of the Earth is 7925.65 miles; the *Polar*, 7899.17; and the *Mean*, 7916.17.

371. Circular Measure. $60'' = 1'$, $60' = 1°$, $90° = 1$ *quadrant.*

A *Geographic, Nautical*, or *Marine Mile* is 1′ of the equator or of a meridian, and hence is 1.1527 *Eng. mi.*, very nearly, a degree being 69.164+ *Eng.* or *Statute miles.*

A *League* is 3 marine miles.

A *Knot* is properly $\frac{1}{120}$ of a marine mile (50.72 *ft.*), but current usage makes it equivalent to a marine mile. Thus the original and proper expression was "Sailing 12 knots," not "Sailing 12 knots *per hour;*" but as sailing "12 knots" was equivalent to sailing 12 *miles* per hour, the expression "Sailing 12 knots per hour" has come into use as meaning "Sailing 12 miles per hour."

A *Fathom* = 6 *ft.* A *Cable's Length* = 120 fathoms.

372. Surveyor's Measure. — A *Chain* = 100 links = 4 *rods.* Hence a link = 7.92 *in.*, 80 *ch.* = 1 *mi.*, and 10 *sq. ch.* = 1 *Acre.*

Wood or ***Stone Measure.***—A *Cord Foot* is 1 *ft.* in length of a pile of 4 *ft.* wood 4 *ft.* high, and hence is 16 *sq. ft.* A *Perch* is $24\frac{3}{4}$ *cu. ft.*

Measures of Weight.

373. There are three varieties of weight in legal use in the United States, viz.: *Avoirdupois, Troy*, and *Apothecaries'*. The *Standard Unit* for all is the *Troy Pound.* The *Government Standard* is a brass weight procured by the American minister at London in 1827, and was a copy of the English Standard Troy Pound. Congress enacted that this brass weight should be the standard, and that it should be reckoned 5760 grains, 7000 of which should constitute the *Avoirdupois* pound. The *Apothecaries' Pound* is the same as the Troy pound.

AVOIRDUPOIS WEIGHT.

16 *oz.* = 1 *lb.*
100 *lb.* = 1 *cwt.*
20 *cwt.* = 1 *T.*

TROY WEIGHT.	APOTHECARIES' WEIGHT.
24 *gr.* = 1 *pwt.*	20 *gr.* = 1 *sc.*, or ℈.
	3 ℈ = 1 *dr.*, or ʒ.
20 *pwt.* = 1 *oz.*	8 ʒ = 1 *oz.*, or ℥.

12 ℥ = 1 *lb.*, or ℔.

In the U. S. Custom-House business 112 *lb.* = 1 *cwt.*; hence 28 *lb.* = 1 *qr.* (*cwt.*), and 2240 *lb.* = 1 *ton*, often called the *Long Ton.* This weight is usually used in weighing coal in large quantities, iron, and iron ore. But for general purposes 100 *lb.* is the legal hundredweight, and 2000 *lb.* the ton.

The word *Cental* is coming into use instead of hundredweight. 100 *lb.* of dry fish is called a *Quintal.*

For Physicians' practices see note after next tables.

Measures of Capacity.

374. There are *two varieties* of measures of capacity in legal use in this country, viz.: *Liquid Measure* and *Dry Measure.* The U. S. *Standard* of Liquid Measure is the *Gallon*, which Congress has enacted shall contain 58372.1754 grains of distilled water at its maximum density (39.83° Fah.*), the barometer at 30 *in.* This makes the liquid gallon very nearly 231 *cu. in.* The U. S. Bushel is 543391.891 *gr.* of distilled water, under the same conditions as above. This makes the bushel (which is the old English Winchester bushel) 2150.42 *cu. in.*, or a cylinder 18½ *in.* in diameter and 8 *in.* deep.

LIQUID MEASURE.	DRY MEASURE.
4 *gi.* = 1 *pt.*	
2 *pt.* = 1 *qt.*	2 *pt.* = 1 *qt.*
4 *qt.* = 1 *gal.*	8 *qt.* = 1 *pk.*, or 2 *gal.*
31½ *gal.* = 1 *bbl.*	4 *pk.* = 1 *bu.*

Barrels are made of various sizes from 30 to 40 or even 56 gallons; but in estimating the capacity of cisterns, vats, etc., 31½ *gal.* is usually considered a barrel. There is no definite measure in use called a hogshead. Any large cask is frequently so called.

Physicians and Apothecaries use a kind of liquid measure of

* This is the temperature as assumed by Prof. Hassler in determining our standards, although the maximum density is now reckoned as at 39.2°.

which the denominations are *Minims* (♏), *Fluid Drachms* (f ʒ), *Fluid Ounces* (f ℥), *Pints*, and *Gallons*. The pint and gallon are the same as the common *Liquid Pint* and *Gallon*, but are designated by the abbreviations (O.) (Latin *octarius*, pint), and *Cong*. (Latin *congius*, gallon. 60 ♏ = 1 f ʒ, 8 f ʒ = 1 f ℥, and 16 f ℥ = 1 (O.).

Physicians in making prescriptions frequently call a minim a *drop*, a fluid drachm a *teaspoonful*, 4 fluid drachms a *tablespoonful*, a fluid ounce 2 *tablespoonfuls*, 4 fluid ounces a *teacupful*, and a pint 4 *teacupfuls*. The physician's abbreviation for drop is *gtt*.

These measures are very indefinite, and in fact are much in excess of what they are called. Thus a drop of most liquids is much more than a *minim*. A common teaspoon holds nearer 90 than 60 drops of water, and we more frequently find teacups that hold ½ a pint than a gill.

℞ is an abbreviation for *recipe*, or take; ā, aa, for equal quantities; *ss*. for *semi*, or half; gr. for grain; P. for *particula*, or little part; P. æq. for equal parts; q. p., as much as you please.

In writing Prescriptions Physicians use the Roman notation, and write the symbol indicating the denomination before the number; thus ʒ ij signifies 2 drachms, ℥ iv is 4 *oz.*, *gr*. v is 5 grains, etc., j being written for final i.

Table showing the Weight of a Bushel of the principal grains and seeds, as established by Law in the several States.

Barley.	Ill., Ind., Ia., Ky., Mich., Minn., Mo., N. C., N. J., Ohio, Wis. **48** *lb.* Mass., Or., Vt. 46 *lb.*; W. T. 45 *lb.*; La. 32 *lb.*; Pa. 47 *lb.*; Cal. 50 *lb.*
Buck-wheat.	Mich., Minn., Or., Wis. 42 *lb.*; Ia., Ill., Ky., Mo. 52 *lb.*; Ind., N. C., N. J. 50 *lb.* Cal. 40 *lb.*; Mass., Vt. 46 *lb.*; N. Y., Pa. 48 *lb.*; Conn. 45 *lb.*
Clover Seed.	Ill., Ind., Ia., Ky., Mich., Minn., Mo., N. Y., Ohio, Or., W. T., Wis. **60** *lb.* N. J. 64 *lb.*
Indian Corn.	Conn., Del., Ind., Ia., Ill., Ky., La., Mass., Mich., Minn., N. J., Ohio, Or., Pa., Vt., W. T., Wis. **56** *lb.* Cal., Mo. 52 *lb.*; N. C. 54 *lb.*; N. Y. 58 *lb.*
Oats.	Cal., Ill., Ind., La., Mich., Minn., N. Y., Ohio, Pa., Vt., Wis. **32** *lb.* Me., Mass., N. C., N. H., N. J. 30 *lb.*; Ia., Mo. 35 *lb.*; W. T. 36 *lb.*; Conn. 28 *lb.*; Ky. 100 *lb.* to 3 *bu.*
Rye.	Conn., Ind., Ia., Ill., Ky., Mass. Mich., Minn., Mo., N. J., N. Y., Ohio, Or., Pa., Vt., W. T., Wis. **56** *lb.* Cal. 54 *lb.*; La. 32 *lb.*
Timothy Seed.	Ill., Ind., Ia., Ky., Mo. **45** *lb.* N. Y. 44 *lb.*; Wis. 46 *lb.*
Wheat.	**60** *lb.* in all except Conn. In Conn. 56 *lb.*

A bushel of coal is usually reckoned 80 *lb.*

Peas, *Beans*, and *Potatoes* are usually weighed at 60 *lb.* to the bushel.

196 *lb.* *Flour* make a barrel. 200 *lb.* *Pork* or *Beef* make a barrel.

In England a *Quarter* of grain is 560 *lb.* or 8 Imperial Bushels.

Measures of Time.

375. The *Standard Unit* of time is the *Mean Solar Day*. A day is in a general sense the time required for the earth to make one revolution on its axis. In astronomic and accurate language a *Sidereal* (star) *Day* is the time between two successive passages of the meridian by the same fixed star.* Such days are of absolutely uniform length. A *Solar* (sun) *Day* is the time between two successive passages of the Sun across the same meridian. This is a little longer than the Sidereal Day, since as the earth revolves on its axis it is going in the same way around the sun, and hence has to make a little more than one revolution to bring the sun to the same meridian. Again, as this motion around the sun is more rapid in our winter than in summer, this difference is greatest in winter and least in summer. The *Mean Solar Day* is the average of all the solar days in a year. From this we see that a clock which keeps uniform time will be sometimes faster and sometimes slower than sun time; thus on Feb. 11, 1876, the sun came to the meridian (*i. e.*, it was noon by the sun) 14 *min.* and 30 *sec.* after 12 by the true clock, and on Nov. 8 it will be noon by the sun at 11 o'clock 43 *min.* 55 *sec.* by a true clock. These are the extreme differences for the year.

The *Mean Solar Day* is divided into 24 equal parts called *Hours*. Of these the *Sidereal* (or true day) contains 23 *hr.* 56 *min.* 4.1 *sec.*

The *Civil Day* is the Mean Solar Day commencing at midnight. The Astronomical Day is the *Sidereal* Day, and is reckoned from one passage of the vernal equinox to another, the hours being counted on from 1 to 24.

TABLE.

$60'' = 1'$; $60' = 1$ *hr.*; 24 *hr.* = 1 *da.*; 7 *da.* = 1 *week.*

The *True* or *Tropical Year* = 365 *da.* 5 *hr.* 48 *min.* 47.8 *sec.* But for practical purposes the ordinary civil year is 365 days, and leap year 366 days.

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* More strictly, the Vernal Equinox.

Every year whose number is divisible by 4, except the centennial years which are not divisible by 400, is a *Leap Year*.

Thus 1840, 1844, 1880, 1876, 1600, 1200, 2000, are leap years. 1551, 1842, 1883, 1500, 1100, 1900, are not leap years.

The reason for *Leap Year* is this: Instead of reckoning the fraction of a day, it is neglected, and a *whole* day is added to the year every 4th year (in general). But as this is a little too much, the centennial years (in general), although they are the 4th years, are reckoned as common years (365 *da.*). But this again is rejecting too many leap years: so that every centennial year which is divisible by 400 is made a leap year. With this correction the error does not amount to a day in 100,000 years.

Formerly the year began on the 25th of March. Russia, and other countries in which the Greek Church is the established Church, adhere to the old style. Their month dates are at this time, therefore, 11 days behind ours; and from the 1st of January to the 25th of March the number which designates their year is one less than that which designates ours.

An interesting question in reference to dates is suggested when we consider that, could we start with the sun in our meridian on any given day, as Monday, May 1st, and pass with the sun around the earth, the day would continue to be to us Monday, May 1st, till we returned to the place of departure, where the people would be calling it Tuesday, May 2d. And in like manner, however long we might be in passing around the earth *from east to west*, we should find our reckoning *one day behind* when we returned to the starting-place. If, on the contrary, we were to go around from *west to east*, we should find our date *one day in advance*. Shipmasters actually change their reckoning at the 180th meridian from Greenwich to accord with these facts. Thus were you going from California to Canton, China, and crossed the 180th meridian on Monday, you would see posted in conspicuous places on the ship an announcement that, having crossed the 180th meridian, the date is changed from Monday to Tuesday. So likewise, if you were returning from Canton to San Francisco, and should cross this meridian on Monday, you would see a notice of change of date to Sunday. Another similar question arises from these facts: It is now Tuesday, April 10, 1876, at Ann Arbor, Mich.; over what part of the earth is it Tuesday, and what day of the week is it over the remainder of the globe? The answer is: It is Tuesday back to the east till we reach the point where it is *midnight* (*i. e.*, as it is now 9 P. M., back

135°), and Tuesday west till we reach the 180th meridian. Between the 180th meridian and longitude 51° 9′ 12″ east it is Wednesday.* At midnight on Monday night at Greenwich, Eng., it is Monday all the way back to the east to the 180th meridian, and Tuesday all the way forward to the west to the same meridian. At the instant the midnight meridian coincides with the 180th, all the earth has the same date day.

The last paragraph has reference to the method of changing the reckoning on shipboard. There is, however, what may be called an *International Date Line*, which differs somewhat from the 180th meridian. This may be traced from Chatham Islands, lat. 44° S., long. 177 W., and running north and west, keeping on the east of New Zealand, New Guinea, and Borneo, running through the Philippines, approaching the China shore near Canton, and then bearing off to the northwest outside the Japanese islands, passes up through Behring Straits, and terminates in the north pole. On the west of this line the date is, in general, just one day in advance of what it is on the east.

Measures of Value, or Money.

376. The *Standard Unit* of U. S. money is the *Gold Dollar*, which is .9 pure gold and .1 alloy of silver and copper, the silver not to exceed .1 of the alloy. The standard dollar of this metal is to weigh 25.8 grains. The other gold coins authorized by the act of 1873 are *Quarter Eagle* (\$2½), *Three Dollar* piece, *Half Eagle* (\$5), *Eagle* (\$10), and *Double Eagle* (\$20). These are of the same fineness of the Standard Dollar, and have weights in the same ratios to the weight of the dollar as their values bear to the value of the same.

The *Silver Coins* authorized by the law of 1874 are the *Trade Dollar*, weight 420 grains, the *Half Dollar*, weight 12½ grams, the *Quarter Dollar*, the *Dime*, and the *Half Dime*, the weight of each of the last three bearing the same ratio to the *Half Dollar* that its value does. The metal is .9 pure silver and .1 copper. It will be observed that the *Trade Dollar* is relatively heavier than the other silver coins.

The *Minor Coins* are 5*c*., 3*c*., and 1*c*. pieces. The 5*c*. and 3*c*. pieces are ¾ copper and ¼ nickel, and weigh respectively 5 grams and 1.944 grams. The 1*c*. is 5% tin and zinc (proportioned at the pleasure of the director of the Mint), and 95% copper, weight 48 grains.

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* This statement and the following refer to the civil day which changes at midnight.

Value of Foreign Coins in U. S. Money (gold) as proclaimed by the Secretary of the Treasury January 1, 1875.

COUNTRY.	UNIT.	METAL.	U. S.
Argentine Republic...	Peso fuerte............	G......	$1.00
Austria..............	Florin................	G......	.45,3
Belgium.............	Franc................	G. & S.	.19,3
Bolivia.....	Dollar.................	G. & S.	.96,5
Brazil...............	Milreis of 1,000 reis....	G......	.54,5
Bogota..............	Peso..................	G......	.91,2
Canada..............	Dollar..	G......	1.00
Central America......	Dollar................	S......	.91,8
Chili................	Peso..................	G......	.91,2
Cuba................	Peso..................	G......	.92,5
Denmark............	Crown................	G......	.26,8
Ecuador.	Dollar................	S......	.91,8
Egypt...............	Pound of 100 piasters..	G......	4.97,4
France..............	Franc................	G. & S.	.19,3
Great Britain........	Pound sterling........	G......	4.86,6½
Greece..............	Drachma.............	G. & S.	.19,3
German Empire......	Mark................	G......	.23,8
Hayti...............	Dollar................	S......	.95,2
India...............	Rupee of 16 annas.....	S......	.43,6
Italy................	Lira..................	G. & S.	.19,3
Japan	Yen............... ...	G......	.99,7
Liberia.............	Dollar................	G......	1.00
Mexico..............	Dollar................	S......	.99,8
Netherlands.........	Florin................	S......	.38,5
Norway.............	Crown................	G......	.26,8
Paraguay.......	Peso	G......	1.00
Peru................	Dollar................	S......	.91,8
Porto Rico..........	Peso	G......	.92,5
Portugal............	Milreis of 1,000 reis....	G......	1.08,4
Russia..............	Rouble of 100 copecks..	S......	.73,4
Sandwich Islands....	Dollar................	G.	1.00
Spain................	Peseta of 100 centimes..	G. & S.	.19,3
Sweden.............	Crown...............	G.	.26,8
Switzerland..........	Franc	G. & S.	.19,3
Tripoli	Mahbub of 20 piasters..	S......	.82,9
Tunis...............	Piaster of 16 caroubs...	S......	.11,8
Turkey..............	Piaster...............	G.	.04,3
U. S. of Colombia....	Peso	S.....	.91,8
Uruguay............	Patacon.............	G.	.94,9

In French currency 100 centimes = 1 franc; 20 francs = 1 Napoleon. In German currency 100 pfennige = 1 mark. A Prussian thaler = 74.6, and a groschen 2½ cents.

The Metric System.

277. This system, originally devised and adopted by the French, makes *The Meter* the fundamental unit. It was designed that the Meter should be $\frac{1}{10000000}$ part of a quadrant of a meridian of the earth. With this design an arc of the meridian, starting from the parallel of Dunkirk in the extreme north of France, and running the entire length of France, and terminating in the parallel of Barcelona in the north of Spain, was measured by Delambre and Méchain, as directed by the French government. From this measurement the whole quadrant was computed, and the Meter established as $\frac{1}{10000000}$ part of it. It is now known that there are irregularities in the form of the earth which would make such measurements give different results when taken in different places, and that the Meter thus established is about $\frac{1}{5000}$ of an inch too short. As established it is 3.2808992 *ft.*, or about 39.37079 *in.* For ordinary purposes it is reckoned 39.37 *in.* The *Standard Unit* of *Measures of Capacity* was made the *Cubic Decimeter* ($\frac{1}{10}$ meter), or 61.02705 *cu. in.* The *Cubic Meter* as a measure of capacity is therefore a *Kiloliter* (1000 liters), The *Standard Unit* of *Weight* was made the *Gram* (Fr. *gramme*), which is the weight of a *Cubic Centimeter* ($\frac{1}{100}$ of a meter) of distilled water at the maximum density (4° centigrade, or 39.2 Fah.). The *Gram*, therefore, is 15.4325 grains.

The *Derivatives* from these units are 10ths, 100ths, and 1000ths for the lower denominations, and 10's, 100's, 1000's, and 10,000's for the higher. The lower are designated by the Latin prefixes *Deci* ($\frac{1}{10}$), *Centi* ($\frac{1}{100}$), and *Milli* ($\frac{1}{1000}$). The higher are designated by the Greek prefixes *Deca* (dek'a) (10), *Hecto* (100), *Kilo* (1000), and *Myria* (10,000).

LINEAR MEASURE.

Metric Denominations and Values.					Equivalents in U. S. Measures.	
Myriameter,	Mm.	=	10,000 meters	=	6.2137	miles.
Kilometer,	Km.	=	1,000 meters	=	0.62137	mile.
Hectometer,	Hm.	=	100 meters	=	328 feet 1 inch.	
Decameter,	Dm.	=	10 meters	=	393.7	inches.
Meter,	M.	=	1 meter	=	39.37	inches.
Decimeter,	dm.	=	$\frac{1}{10}$ of a meter	=	3.937	inches.
Centimeter,	cm.	=	$\frac{1}{100}$ of a meter	=	0.3937	of an inch.
Millimeter,	mm.	=	$\frac{1}{1000}$ of a meter	=	0.0394	of an inch.

For minute measurements, as in microscopy, the unit ordinarily spoken of is the *Millimeter* (= about .04 *in.*, or $\frac{1}{25}$). For such purposes as we use the inch the *Centimeter* is used (= about $\frac{2}{5}$ *in.*). For such purposes as we use the rod, yard, and foot the *Meter* is used. For such purposes as we use the mile the *Kilometer* is used (= about $\frac{5}{8}$ *mi.*).

Measures of Surface.

Metric Denominations and Values.			Equivalents in U. S. Measures.
Hectare,	Ha.	= 10,000 sq. meters	= 2.471 acres.
Are,	A.	= 100 sq. meters	= 119.6 sq. yards.
Centiare,	C.	= 1 sq. meter	= 1550 square inches.

The Hectare, Are, and Centiare are the units of land measure. For the measurements of other surfaces, squares whose sides are the multiples and submultiples of the meter are used.

Measures of Solids.

Metric Denominations and Values.			Equivalents in U. S. Measures.
Decastere,	Ds.	= 10 cubic meters	= 13.079 cubic yards.
Stere,	S.	= 1 cubic meter	= .2759 of a cord of wood.
Decistere,	Ds.	= 100 cnbic dm.	= 3.53144 cubic feet.

The Stere, Decastere, and Decistere are used only in measuring fire-wood and timber; for measuring other volumes the cubes of the linear measures are used, as the *cubic meter*, *centimeter*, etc.

Measures of Capacity.

Metric Denominations and Values.			Equivalents in U. S. Measures.	
Names.	No. of Liters.	Cubic Measure.	Dry Measure.	Liquid or Wine Measure.
Kiloliter, Kl. =	1,000 =	cubic meter or stere	= 1.308 cubic yards	= 264.17 gallons.
Hectoliter, Hl. =	100 =	$\frac{1}{10}$ of a cubic meter	= 2 bu. 3.35 pk.	= 26.417 gallons.
Decaliter, Dl. =	10 =	10 cubic decimeters	= 9.08 quarts	= 2.6417 gallons.
Liter, L. =	1 =	1 cubic decimeter	= 0.908 of a quart	= 1.0567 quarts.
Deciliter, dl. =	$\frac{1}{10}$ =	$\frac{1}{10}$ of a cu. decimeter	= 6.102 cubic in.	= 0.845 of a gill.
Centiliter, cl. =	$\frac{1}{100}$ =	10 cubic centimeters	= .6102 " "	= 0.338 of a fl. oz.
Milliliter, ml. =	$\frac{1}{1000}$ =	1 cubic centimeter	= .0610+ " "	= 0.27 of a fl. drm.

The *liter* is commonly used in measuring milk, wine, etc., in moderate quantities. For minute quantities, the *centiliter* and *milliliter* are employed; and for large quantities, the *decaliter*. For

measuring grain, etc., the *hectoliter*, which is equal to 2.8375 bushels, is commonly used. Instead of the *kiloliter* and *milliliter*, it is customary to use *cubic meters* and *cubic centimeters*, which are their equals.

MEASURES OF WEIGHT.

Metric Denomination.		No. of Grams.	Weight of Volume of Water at (39.2° F.)	Equivalents in U. S. Measures.
(Millier) Tonneau,	T.	=1,000,000=	1 cubic meter	=2204.6 lbs. Avoir. W.
Quintal,	Q.	= 100,000=	1 hectoliter	=220.46 lbs. " "
Myriagram,	Mg.	= 10,000=	10 liters	=22.046 lbs. " "
Kilogram,	Kg.	= 1,000=	1 liter	=2.2046 lbs. " "
Hectogram,	Hg.	= 100=	1 deciliter	=3.5274 oz. " "
Decagram,	Dg.	= 10=	10 cubic centimeters	=0.3527 oz. " "
Gram,	G.	= 1=	1 cubic centimeter	=15.432 grs. Troy W.
Decigram,	dg.	= $\frac{1}{10}$=	$\frac{1}{10}$ of a cu. centimeter	=1.5432 grs.
Centigram,	cg.	= $\frac{1}{100}$=	10 cubic millimeters	=0.1543 of a gr.
Milligram,	mg.	= $\frac{1}{1000}$=	1 cubic millimeter	=0.0154 of a gr.

The common unit of weight for groceries, etc., is the Kilogram, usually called a "Kilo." It will be seen to be about $2\frac{1}{5}$ *lb.* The *Tonneau*, usually called the "Ton," is used as we use the *Ton*. It is a near equivalent to our long ton. The gram and its subdivisions are used as we use Troy and Apothecaries' Weights.

MEASURES OF ANGLES.

Metric Denominations and Values.					Equivalents in U. S. Measures.
Circle,	C.	=	400 grades	=	1 circle, or 360°.
Quadrant,	Q.	=	100 grades	=	1 quadrant, or 90°.
Grade,	*g.*	=	1 grade	=	54 minutes.
Minute,	′	=	$\frac{1}{100}$ of a *g.*	=	32.4 seconds.
Second,	″	=	$\frac{1}{10000}$ of a *g.*	=	.324 of a second.

ANSWERS.

Pages 12, 13. *1.* 105, 4000. *2.* 968000, 182304. *3.* 1375, 1, 100, 9. *6.* abx. *7.* $26am$. ***(36.)*** *Ex.* $3amxy$, $225abcd$, $abcd$, and ab are monomials; $2ax-3b$, $c-d$, $a+m$, $a-b$, $x+y$, and $10a+3xy$ are binomials; $c-xy+ax$ is a trinomial; and all except the monomials may be called polynomials, although this term is usually applied to expressions composed of 3 or more terms.

Pages 24-28. *5.* $\frac{40}{88}$, $\frac{20}{65}$. *6.* 8. *7.* 59, 58, 121, 1955. *9.* $\frac{21}{8}$. *11.* $\frac{68}{5}$, $\frac{710}{7}$, $\frac{752}{13}$. *14.* $\frac{32}{12}$, $\frac{36}{12}$, $\frac{9}{12}$, $\frac{18}{12}$; $\frac{64}{24}$, $\frac{72}{24}$, $\frac{18}{24}$, $\frac{36}{24}$; $\frac{128}{48}$, $\frac{144}{48}$, $\frac{36}{48}$, $\frac{72}{48}$. *17.* .75, .35, .275, .16, .06. *18.* .666+, .8333+, .571428571+, .5454+, .230769230769+. *19.* $.\dot{1}9047\dot{6}$, $.0\dot{6}$, $.\dot{1}37254901960784\dot{3}$, $.\dot{1}4285\dot{7}$, $.\dot{4}2857\dot{1}$, $.\dot{7}05882352941176\dot{4}$. *21.* $\frac{12}{25}$, $\frac{7}{200}$, $\frac{3}{400}$, $\frac{1}{2}$, $\frac{4}{5}$, $\frac{3}{25}$, $\frac{173}{400}$, $\frac{1}{16}$. *22.* $\frac{3}{2}$, $\frac{9}{4}$, $\frac{1691}{200}$, $\frac{85}{8}$, $\frac{15}{2}$, $\frac{5201}{40}$, $\frac{2403}{80}$. *24.* $\frac{2}{9}$, $\frac{1}{3}$, $\frac{4}{9}$, $\frac{5}{9}$, $\frac{2}{3}$, $\frac{8}{9}$, 1. *25.* $\frac{1}{90}$, $\frac{7}{90}$, $\frac{4}{45}$, $\frac{2}{45}$, $\frac{1}{18}$, $\frac{1}{10}$. *26.* $\frac{1}{900}$, $\frac{1}{150}$, $\frac{1}{100}$, $\frac{7}{90000}$, $\frac{1}{1800}$, $\frac{1}{2250}$. *27.* $\frac{1}{99}$, $\frac{2}{99}$, $\frac{3}{33}$, $\frac{4}{999}$, $\frac{5}{9999}$, $\frac{8}{999}$, $\frac{7}{99999}$. *28.* $\frac{335}{999}$, $\frac{43}{99}$, $\frac{1714}{3333}$, $\frac{9}{11}$, $\frac{25}{33}$, $\frac{3}{111}$, $\frac{326}{909}$. *29.* $\frac{3}{110}$, $\frac{3}{1100}$, $\frac{3}{11000}$, $\frac{29}{6660}$, $\frac{29}{66600}$, $\frac{149}{49950}$, $\frac{428}{499500}$. *30.* $\frac{49}{150}$, $\frac{1753}{3300}$, *, $\frac{12049}{39960}$, $\frac{17}{45}$; $\frac{4086009}{9999000}$, $\frac{280993}{99990000}$, $\frac{689}{1125}$.

Pages 29-31. *2.* 48 *oz.* *3.* 12 *oz.* *6.* 1087*d.* *7.* 36 *far.* *8.* $\frac{1}{2}$ *far.* *12.* 1609.34+ *M.* *13.* 9.072+ *k.* *14.* 25900.4+ *ares.* *15.* $\frac{3}{16}$ *ft.* *16.* 11125.79− *bbls.* *17.* 1.6093+ *km.* *18.* 3.785+ *L.* *19.* .276− *cd.* *20.* .652$\frac{7}{8}$ *yr.* *21.* $\frac{3}{65}$ *mo.* *22.* 24$\frac{1}{3}$ *da.*, or 24 *da.* *23.* 2.76− *cd.* *24.* $\frac{1}{378}$ *bbl.* *25.* 15.24+ *mm.* *26.* 371.82+ *yd.* *27.* 21504 *cu. in.* *28.* 5.03− *M.* *29.* 2.471+ *A.* *30.* 2.8375 *bush.* *31.* .875 *gal.* *32.* 178.75+ *fr.* *33.* 342.23− *M.* *34.* $579. *35.* 12208 *cu. in.* *36.* 1.6535+ *in.* *37.* 3$\frac{11}{288}$ *lb.* *38.* 2527.554 *in.* *39.* 11.16+ *A.* *40.* 10 *oz.* 18 *pwt.* 22.56 *gr.* *41.* 562.542*c.*, or 5625.42*m.* *42.* 133 *rd.* 5 *ft.* 6 *in.* *43.* 112 *cu. ft.* *44.* $\frac{1}{480}$ *mi.* *45.* 9.3+ *bush.* *46.* 6.58+ *oz.* *47.* .00088+ *oz.* *48.* 1.555+*G.* *49.* 9.84$\frac{1}{4}$ *in.* *50.* 11720″. *51.* 3°.311$\frac{2}{3}$. *52.* 5181.34+ *fr.*; 4201.68+*M.* *53.* 20°*C.* *54.* 71$\frac{3}{5}$° *Fah.* *55.* 36$\frac{2}{3}$° *C.* *56.* 161°.6 *Fah.*

Pages 35-37. *3.* 7430.688. *9.* 4 *lb.* 10 *oz.* 2 *pwt.* 12 *gr.* *10.* 17.56561 *M.*, or 57.6298$\frac{3}{8}$ *ft.*, or 691.5584 *in.* *11.* 237.9 *gal.*, or 25.555+ *bush.* *12.* 575.5474+ *cu. ft.*, or 4$\frac{1}{2}$ *cd.*, very nearly.

13. \$1077.524. *14.* 126485 *ft.* *15.* $4443\frac{3}{4}$ *gr.*, or $5373\frac{3}{4}$ *gr.* *16.* 20.67+ *sq. ft.* *17.* $2\frac{513}{560}$. *18.* 351.02 *in.* *19.* 22.875. *20.* 7.041 *mi.*, nearly.

Pages 37-39. *2.* $19a$. *8.* $15cd+13ax+25cy+36ay$. *9* $15x^2$. *12.* $44a^2+93a^3+29x$.

Pages 42-45. *4.* 2.1. *5.* 21. *7.* 24 *yd.* 1 *ft.* 4 *in.*; 244 *yd.* 1 *ft.* 4 *in.*; 2444 *yd.* 1 *ft.* 4 *in.* *12.* 4011.32. *15.* 24.208. *16.* $58\frac{4}{21}$. *17.* 37 *bbl.* 29 *gal.* $\frac{1}{4}$ *qt.* *18.* 3 *lb.* 5 *pwt.* 15 *gr.* *19.* 22 *cd.* $18\frac{4}{5}$ *cu. ft.* *20.* 37° 27′ 5.6″. *21.* 18 *A.* 117 *sq. rd.* 108.4 *sq. ft.* *22.* .01944; 32.42592. *23.* $.00\frac{1}{3}$; $\frac{1}{128}$. *24.* $5.4\frac{3}{43}$; $\frac{81}{6020}$. *25.* 1 *mi.* 3840 *ft.*; 456 *ft.*

Pages 46-48. *1.* 16; 64; 100; $\frac{1}{4}$; .009. *2.* 36; 216. *4.* 576; 1296. *5.* 2601; 36. *6.* 15625; 103823. *7.* 175616000; 64; 27. *14.* $a^3+3a^2b+3ab^2+b^3$. *16.* $2a^4x^2+5a^2bxy+2b^2y^2$. *17.* $x^2+2xy+y^2$. *20.* $9a^4y+6a^2by^2+3a^2cy$. *21.* $x^5+x^4+x^3+x^2+x+1$.

Pages 49-54. *1.* 216, 7776. *4.* 1849; 362404; 27.04; $76\frac{9}{16}$; $28\frac{4}{9}$; .001156; 1006.1584; $\frac{25}{36}$; $\frac{49}{121}$; $6\frac{1}{4}$; 100; 10000; 1000000, 100000000. *5.* 592704; 2299.968; $3\frac{3}{8}$; .125; $\frac{8}{27}$ .000512; 27543608; 1000; 1000000; 1000000000. *7.* 2744; 12326391; 32768000. *10.* $a^2+2ab+b^2$; $4a^2x^2+12abx+9b^2$; $1+2y+y^2$; $4+4cx^2+c^2x^4$. *11.* $a^3+3a^2b+3ab^2+b^3$; $64a^3+96a^2b^2+48ab^4+8b^6$; $x^6+3x^4y^2+3x^2y^4+y^6$; $1+3x+3x^2+x^3$. *15.* 266635-235464391245607.

Pages 57-59. *7.* 2751.065 *in.* *8.* 6 *gal.* 2 *qt.*; 4 *gal.* 2 *qt.* *11.* \$3.324; $\frac{13}{85}$. *15.* 2 *cd.* 91 *cu. ft.* 420 *cu. in.* *16.* \$875. (***102.***) *1.* 10° 48′ 2″; *2.* 73° 42′ 38″. *3.* 61° 18′ 3″. *4.* 58°.

Pages 59-67. *2.* $3ay$. *3.* $4a^2x$; $5by^2$. *7.* $3a-4b$. *9.* x^3-x^2; x^2-x; $a-1$; $1-x$. (***104.***) *2.* \$$5ax$. *4.* $8x^2+7y^3-6cy$. (***105.***). *12.* $9ax+10$. *13.* $4x+10a$. *14.* $15+10z$. *15.* $-2x$. *17.* $8a-6by$. *18.* $16-7x-6a$. *19.* $11a^2-16b$. (***106.***) *1.* $-4ax+3by-x^2$. *2.* $-7c^2+4ax^2$. *4.* $-5ax$; $+3ay$; $+4a^2y$. *5.* $-7ax+3a^2y$. (***107.***). *8.* $9ax-3b^2y-2c$. *9.* $8ab-17a^2z^2-4a^3y$. *10.* $5a-3b$. *11.* $5-4xy$. (***108.***) *6.* $1-2r+r^2$; $1-r^2$. *7.* $1+6r+15r^2+20r^3+15r^4+6r^5+r^6$; $1-5r+10r^2-10r^3+5r^4-r^5$. *Ex. 1.* $a(a-b)=a^2-ab$; $(a-b)(a+b)=a^2-b^2$; $(a+c)^2=a^2+2ac+c^2$.

Pages 71, 72. *2.* 8 *bbl.* 14 *gal.* 3 *qt.* $1\frac{5}{9}$ *pt.* *5.* 7.25+. *7.* 6.477+; $17\frac{1}{2}$. *8.* 8.69+. *12.* 9.

Pages 77, 78. *7.* $\frac{3a}{b}$. *8.* $-\frac{4a}{x}$. *9.* $-\frac{6ax}{b^2}$. *10.* $\frac{c^2}{x^2}$. *11.* $\frac{a}{b}$. *12.* $-\frac{4}{x}$. *13.* $\frac{m}{4nx^2}$. *14.* $\frac{16b^2}{mn}$. *15.* $\frac{1}{3abx}$. *16.* $\frac{4x}{7ay}$. *17.* $\frac{8}{9x^2y}$. *18.* $\frac{2a}{7x}$. *23.* $2z+3y$. *24.* $5a-9$. *25.* $2x-1$. *26.* $-a+d+x$. *27.* $1+2x+a-5$. *28.* $c+4x-3a$. *29.* $40a^2b+60ab-17$. *30.* $-3ab+2x^2-d^2$. *31.* $4x+3x^2+2x-1$.

Pages 80-110. To give answers to numerical examples in Evolution would strip the exercises of most of their value; and the same may be said of examples in Factoring.

Pages 117-121. *4.* 3. *5.* 12. *6.* 2. *7.* 42. *8.* 7. *12.* $\frac{3ab}{m-n}$. *13.* $\frac{5am-10an}{2b}$. *16.* 36. *17.* 9. *18.* 120. *19.* 60. *20.* 64. *23.* $\frac{100}{1-r}$. *24.* 6. *28.* 8. *29.* 2. *33.* $+2$, and -2. *34.* $+4$, and -4. *35.* $+5$, and -5.

Page 124. *13.* $A=p(1+rt)$; $r=\frac{i}{pt}$; $r=\frac{A-p}{pt}$; $t=\frac{A-p}{pr}$; $t=\frac{i}{pr}$. *14.* $s=\frac{1}{2}n\left\{2a+(n-1)d\right\}$; $d=\frac{l-a}{n-1}$; $n=\frac{l-a}{d}+1$; $s=\frac{l+a}{2}+\frac{l^2-a^2}{2d}$; $n=\frac{2s}{l+a}$; $a=\frac{2s}{n}-l$; $s=\frac{1}{2}n\left\{2l-(n-1)d\right\}$.

Pages 129-132. *2.* \$132.50. *3.* \$219.375. *5.* 42 men. *6.* $12\frac{6}{7}$ *da.* *8.* \$2.625. *9.* 81 men. *10.* $6\frac{2}{3}$ *da.* *11.* 351 *bbl.* *12.* \$1.20. *13.* $7\frac{1}{5}$ *hr.* *14.* $7\frac{1}{8}$ *yd.* *16.* 5 *cd.* 102– *cu. ft.* *17.* 34 *A.* ***(206.)*** *1.* 6; 9; 7.416+; .5773+. *2.* $2\frac{2}{7}$; 18; $5\frac{1}{3}$; 9; $2\frac{2}{3}$; $\frac{24}{25}$.

Page 134. *6.* 24 men.

Page 135. *2.* 15 and 20; 10 and 25; 5 and 30. *3.* $\frac{15}{41}$, $\frac{20}{41}$, and $\frac{6}{41}$. *4.* $1\frac{1}{6}$, $1\frac{17}{18}$, $3\frac{1}{9}$, and $\frac{7}{9}$. *5.* $714\frac{2}{7}$, $1428\frac{4}{7}$, and $2857\frac{1}{7}$. *6.* $26\frac{2}{3}$, $23\frac{1}{3}$, $33\frac{1}{3}$, and $16\frac{2}{3}$.

Pages 137-141. *1.* 26, 124. *2.* 23, 119. *3.* 64, 442. *4.* 99, 2500. *5.* 70, 1260. *6.* 127, 2077. *8.* 48, 1170. *10.* $8\frac{2}{3}$, 82; $1\frac{1}{3}$, 38. ***(217.)*** *2.* 12. *3.* -5. *4.* 41. ***(222.)*** *1.* 5, 16. *2.* 186, -2. *3.* $n=17$, $d=1$; $\therefore$ 1, 2, 3, 4, etc., to 17 is the series. *4.* $d=3$; $\therefore$ 20, 23, 26, etc., to 59 is the series.

5. $d = \frac{3}{28}$; $\therefore$ 3, $3\frac{3}{28}$, $3\frac{6}{28}$, $3\frac{9}{28}$, $3\frac{12}{28}$, $3\frac{15}{28}$, $3\frac{18}{28}$, $3\frac{21}{28}$, $3\frac{24}{28}$, $3\frac{27}{28}$, $4\frac{2}{28}$, $4\frac{5}{28}$, $4\frac{8}{28}$, $4\frac{11}{28}$, $4\frac{1}{2}$ is the series. *6.* 1020 *mi.*, 20 *mi.* *7.* $\frac{1}{4}$ *mi.* *8.* 300.

Pages 143-146. *1.* 640, 1275. *3.* 2186. *4.* $\frac{29}{128}$, $14\frac{35}{128}$. ***(228.)*** *2.* 2. *3.* $16\frac{1}{3}$. *5.* $\frac{2}{9}$; $\frac{1}{33}$; $\frac{1}{30}$; $\frac{5}{198}$; $\frac{25}{999}$. ***(229.)*** *1.* 1860040. *2.* 729. *3.* 2.

Pages 152-155. Many of these examples reciprocally answer each other. *12.* 800. ***(242.)*** *10.* $-\frac{5}{4}$, $-\frac{5}{6}$. *13.* 50%; $33\frac{1}{3}$%; 40%; 75%; 150%; 200%. *14.* $\frac{1}{3}$; $\frac{1}{2}$; $\frac{1}{3}$; the whole; twice; $\frac{3}{4}$; $\frac{1}{10}$. *15.* 8%; 80%; 6%; $6\frac{9}{11}$%; $12\frac{1}{2}$%.

Pages 156-159. *1.* \$4.20 *pr. yd.* *2.* $11\frac{1}{3}$*c.*; $12\frac{1}{2}$*c.*; 11*c.* *8.* \$20.28. *9.* $16\frac{2}{3}$%. *10.* \$26.66$\frac{2}{3}$. *11.* $1\frac{7}{8}$*c.*; \$1; $5\frac{3}{4}$*c.*; \$5; \$2.82. *12.* \$5200. *14.* $33\frac{1}{3}$%. *15.* 20%. *16.* 25%. *17.* $\frac{1}{20}$; $\frac{1}{10}$; $\frac{1}{5}$; $\frac{1}{4}$; $\frac{1}{2}$; cost must be doubled; $\frac{1}{8}$. *18.* \$1.15. *19.* $26\frac{1}{4}$*c.*; $29\frac{1}{4}$*c.*; $24\frac{3}{4}$*c.* *20.* 16%. *21.* \$5.12$\frac{1}{2}$. *22.* 24*c.* *23.* $14\frac{1}{4}$*c.* *24.* $23\frac{61}{193}$%, or about $23\frac{1}{3}$%.

Pages 160, 161. *1.* \$7500. *2.* \$3578.75. *3.* \$3425.93–. *4.* \$71677.50. *5.* \$1484.375. *6.* \$8544.634. *7.* \$5467.742, \$182.258+.

Pages 163-166. *1.* $1\frac{4}{15}$%; \$126.66$\frac{2}{3}$; \$27.86$\frac{2}{3}$. *2.* \$18750; \$937.50; \$17812.50. *3.* .434074%. *4.* (1st) 1.434023 mills. (2d) \$1.27; \$0.20; \$0.18; \$5.74. (3d) 43*c.*; 7*c.*; 17*c.*; 6*c.*; 4*c.*; 7*c.* (4th) $\frac{1}{20}$ mill; .4762– mills. *6.* 1.620466+ cents. *8.* \$109.06; \$10.26; \$345.76. *9.* \$7713.83; \$28.59; \$5.69; \$11570.74+. *10.* \$7.37. *11.* (1st) 1.5583 mills. (2d) \$7.79; 69 cents.

Page 168. *7.* \$626.40.

Pages 174-177. *8.* \$29.68. *9.* \$93.15. *10.* \$14.36. *11.* \$342. *12.* \$340.10. *13.* 2.3 *yr.* *14.* 1 *yr.* 10 *mo.* 12 *da.* *15.* 7 *mo.* 24 *da.* *16.* $11\frac{152}{167}$%. *17.* $7\frac{2}{5}$%. *18.* 8%. *19.* \$64.65. *20.* \$815.90. *26.* \$21.41. *27.* \$83.92. *28.* \$88.70. *29.* \$205.29. ***(265.)*** *2.* \$21.86. *3.* \$521.67.

Pages 179-183. *4.* \$515.46; \$643.08. *5.* \$167.63; \$126.43; \$51; \$306.88; \$568.03; \$386.34; \$601.58; \$473.16. *6.* \$3941.57; \$4428.62; \$4302.39; \$6908.38. *7.* \$132.66; \$134.12. *8.* \$30.79; \$47.17. *9.* \$2816.23; \$2929.15; \$2761.22. *10.* \$83.24; \$130.21; \$133.43. *14.* 9 *yr.* 2 *da.*; 8 *yr.* 15 *da.*; 7 *yr.* 96 *da.* *15.* 5 *yr.* *16.* 8 *yr.* 111 *da.* *17.* 2 *yr.* 3 *mo.* *19.* 8%. ***(270.)*** *2.* \$229.92. *3.* \$666.11. *4.* \$380.77. *5.* \$245.24. *6.* \$5.255.

Page 187. *5.* 5254.44\frac{4}{9}$, or $5253.15+.

Page 195. *3.* $301.93.

Pages 198-200. *3.* $447.97; $445.93; $440.83. *4.* $352.22+; $153.93; $756.53. *5.* $302.74. *6.* $378. *7.* 10%, very nearly. *8.* 34+ *da.* *9.* 79 *da.* *10.* $171.98. *12.* $98.22; Jan. 12, 1877. *13.* $254.53. *14.* $207.24. *15.* $398.79.

Pages 201-211. *3.* $328.27. *6.* $300; $325; $371.43. *7.* $230.77; $250; $285.71; $300. *18.* $816.32.

Page 219. *13.* 9$\frac{11}{221}$%.

Pages 223-227. *1.* $2502.50. *2.* $1501.25. *3.* $1824.088. *4.* $2497.502. *5.* $2500. *8.* 1.01\frac{10}{73}$; $758.55; $86.49, with stamp. *9.* $.97$\frac{351}{730}$, or practically $.98; $487.42; $179.63, with stamp. *10.* 101$\frac{87}{292}$ (101$\frac{1}{3}$)*%; 100$\frac{259}{292}$ (101)%; 100$\frac{139}{292}$ (100$\frac{1}{2}$)%; 101$\frac{167}{292}$ (101$\frac{1}{2}$)%. *11.* 98$\frac{84}{365}$ (98$\frac{1}{10}$)%; 97$\frac{264}{365}$ (97$\frac{7}{10}$)%; 97$\frac{129}{365}$ (97$\frac{1}{3}$)%; 98$\frac{124}{365}$ (98$\frac{1}{3}$)%. *14.* $1250. *15.* $\frac{3}{40}$%; $2500. *16.* $957.21, with stamp.

Pages 229, 230. *4.* $2245.73. *5.* £912 10*s.* 9$\frac{3}{4}$*d.* *6.* $\frac{155}{198}$, or about $\frac{4}{5}$%; 4$\frac{19}{39}$ (4$\frac{1}{2}$)%; $\frac{2700}{9788}$%, or a little more than $\frac{1}{4}$%. *7.* 4.96+; 5.092; 97.58. *11.* 39$\frac{11}{51}$*c.*, or about 39$\frac{1}{5}$*c.*

Page 235. *10.* 6 *mo.*

Pages 238, 239. *3.* $316.47. *4.* $1323.61. *5.* $9303.01.

Pages 240, 241. *2.* $\frac{3}{10}$%. *3.* $11648. *4.* $6480. *5.* $4663.21. 6. .0$\frac{8}{9}$%. *7.* $1734.87. *8.* $3750. *9.* 2%. *10.* 1149.16\frac{2}{3}$.

Pages 245-248. *7.* $3072.28. *8.* $3571.93−, assuming that the last deposit is made on the 20th birthday. *9.* $13576.05; $9299.45; $11638; P_w = $2537.85; $3553.90; $6231.10. *3.* $62500.

Page 249. *5.* $4420.17. *2.* $6954.85.

Page 252. *3.* $\frac{68842}{69804}$; $\frac{57917}{69804}$; $\frac{88569}{69804}$; $\frac{14474}{69804}$.

Pages 255, 256. *7.* $1234.39. *2.* $\frac{5531661}{5660406}$, nearly; $\frac{5181284}{5365257}$, nearly. *3.* $\frac{8510010318}{8851465850}$. *6.* $215.89; $215.94.

* The numbers in the parentheses are the practical approximations.

Pages 266, 267. *2.* A \$1050, B and C each \$750, D \$1950. *4.* A \$200, B \$400, C \$800, D \$1400. *5.* \$22.04, \$13.57, \$28.49. *6.* \$13750; \$35.20.

Pages 267, 268. *2.* A \$24.193, B \$51.613, C \$24.193. *3.* \$10.80, \$19.20, \$8.40. *4.* \$73.13, \$257.44, \$284.35, \$585.08. *5.* A \$501.40, B \$376.05. *6.* \$5700 A's, \$3760 B's, \$1340 C's. *7.* \$2250 B's, \$5625 C's. *8.* A's \$4332, B's \$4075.50, C's \$5937.50. *9.* B's stock \$1600, C's time 15 *mo.*

Pages 269-276. *1.* 70 *A.* *2.* 502.656 *A.*; 3.1416 *mi.* *3.* 122° 26′ 45″. *4.* 207.114+ *mi.* *6.* 15°; 34°.28+; 9817.5. *7.* 51 *m.* 9 *sec.*+ slow; 22 *m.* 9 *sec.*− fast. *8.* 95° 4′ 55.8″ W. *9.* 20th. *10.* 18 by 24 *ft.* *11.* 41$\frac{1}{7}$ *ft.* *12.* 40 *m.* 7 *sec.* after 12, nearly. *13.* 90th West; 90th East; 180th. *14.* 36 *min.*; A 5184 *ft.*, B 5616 *ft.* *15.* \$159.37$\frac{1}{2}$. *16.* 26$\frac{6}{7}$ *ft.* *17.* \$140. *18.* 97° 2′ E.; 7° 2′ E.; 172° 58′ W. *19.* 5681. *20.* 1 *A.* 10$\frac{1}{3}$ *sq. rd.* *21.* 9 o'clock 4 *min.* 36.4+ *sec.* *22.* 12 *ft.* .996 *in.* *23.* 6 *rd.* 3 *ft.* *24.* 73.11 *bbl.* *25.* 197,061,258.24 *sq. mi.* *26.* .5236 *cu. ft.*; 260, 120, 860, 876.8 *cu. mi.* *27.* 47.124 *sq. ft.* *28.* 29.452 *A.* *29.* 150.8 *cu. ft.*, nearly. *30.* 77.59+ *bbl.* *31.* \$664. *32.* 2.503+ *qt*, or about $\frac{1}{4}$ *pt.* less than 1 *gal.* *33.* 200.43+ *sq. ft.* *34.* 34.37− *gal.*; 49.48− *gal.* *35.* 25.45+ *in.* *36.* 21.2+ *in.* *38.* 42$\frac{2}{3}$ *cu. ft.* *40.* 23.08+ *in.* on an edge. *41.* 6.545 *ft.* *42.* 17° 11′ 19.3″. *43.* 62.17+ *bbl.* *44.* 8.818$\frac{1}{3}$ *A.* *45.* 117513404+ *bbl.* *46.* 13,983,242.4+ *bbl.*

Pages 276, 277. *2.* 1152; 960; 1280. *3.* 1000; 2400. *4.* 48; 72; 64. *5.* 121; 83+; 136+; 227−

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